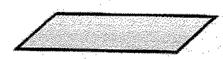
Intersection of Three Planes

When investigating the intersection of three planes, many possibilities may occur.

Type #1 – The Normal Vectors of All Three Planes are Parallel

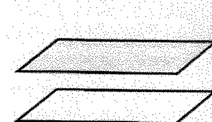
Example
Three Planes are Identical



π_1: $x+y+z+2=0$
π_2: $2x+2y+2z+4=0$
π_3: $3x+3y+3z+6=0$

The three equations are scalar multiples of each other.
Specifically, $\pi_2 = 2\pi_1$ and $\pi_3 = 3\pi_1$
$\therefore$ three planes are identical

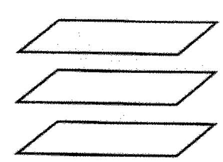
Example
Two Planes are Identical and Other Plane is Parallel



π_1: $x+3y+5z-10=0$
π_2: $2x+6y+10z-18=0$
π_3: $x+3y+5z-9=0$

The three normals are scalar multiples of each other ($\vec{n}_1 = n_3$ and $\vec{n}_2 = 2\vec{n}_1$)
Also, $\pi_2 = 2\pi_3$
$\therefore$ three planes are parallel, two of which are identical (π_2 and π_3)

Example
Three Planes are Parallel and None are Identical

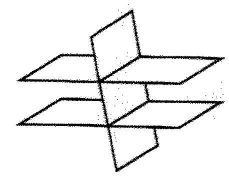


π_1: $x+y+z+2=0$
π_2: $2x+2y+2z+5=0$
π_3: $x+y+z-3=0$

The three normals are scalar multiples of each other ($\vec{n}_2 = 2\vec{n}_1$ and $\vec{n}_3 = \vec{n}_1$)
None of the equations are scalar multiples of each other
$\therefore$ three planes are parallel, but none are identical

Type #2 – Only Two of the Normal Vectors are Parallel

Example
Two Planes are Parallel and Distinct; the Other Plane is Not Parallel



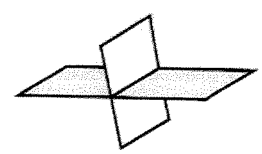
π_1: $3x + 5y - 2z - 4 = 0$
π_2: $4x - 7y + 6z + 11 = 0$
π_3: $6x + 10y - 4z + 1 = 0$

$\vec{n}_3 = 2\vec{n}_1$ $\therefore$ π_1 and π_3 are parallel (but not identical)

π_2 is not parallel to π_1 and π_3

$\therefore$ π_2 intersects π_1 and π_3 along two distinct parallel lines.

Example
Two Planes are Identical; the Other Plane is Not Parallel



π_1: $3x + 5y - 2z - 4 = 0$ ①
π_2: $6x + 10y - 4z - 8 = 0$ ②
π_3: $4x - 7y + 6z + 11 = 0$ ③

$\pi_2 = 2\pi_1$ $\therefore$ π_1 and π_2 are identical

π_3 is not parallel to π_1 and π_2

Find intersection of π_1 and π_3:

① $\times 3$ + ③ $13x + 8y - 1 = 0$

Let $\boxed{x = t}$

$\therefore$ $13(t) + 8y - 1 = 0$

$\boxed{y = \frac{1}{8} - \frac{13}{8}t}$

Sub. in ① to find z:

$3(t) + 5\left(\frac{1}{8} - \frac{13}{8}t\right) - 2z - 4 = 0$

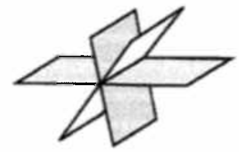
$\boxed{z = -\frac{27}{16} - \frac{41}{16}t}$

$\therefore$ three planes intersect along line with parametric equations

$\boxed{\begin{aligned} x &= t \\ y &= \tfrac{1}{8} - \tfrac{13}{8}t \\ z &= -\tfrac{27}{16} - \tfrac{41}{16}t \end{aligned}}$ $\left(\text{or} \quad \begin{aligned} x &= 16t \\ y &= \tfrac{1}{8} - 26t \\ z &= -\tfrac{27}{16} - 41t \end{aligned} \right)$

Type #3 – None of the Normal Vectors are Parallel

Example
Three Planes Intersect in a Single Line



π_1: $x+2y-3z-4=0$ ①
π_2: $5x+y+2z-7=0$ ②
π_3: $7x+5y-4z-15=0$ ③

None of the planes are parallel

①×5 − ② $9y-17z-13=0$ ④
①×7 − ③ $9y-17z-13=0$ ⑤

Subtract to eliminate y

$0z+0=0$

$0z=0$

∞-many solutions

$\boxed{\text{Let } z=t}$

Sub. in ④ to find y

$9y-17(t)-13=0$

$$\boxed{y=\frac{13}{9}+\frac{17}{9}t}$$

Sub. in ① to find x

$x+2\left(\frac{13}{9}+\frac{17}{9}t\right)-3(t)-4=0$

$$\boxed{x=\frac{10}{9}-\frac{7}{9}t}$$

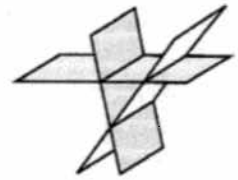
∴ three planes intersect along a line with parametric equations

$$\boxed{\begin{aligned} x&=\frac{10}{9}-\frac{7}{9}t\\ y&=\frac{13}{9}+\frac{17}{9}t\\ z&=t\end{aligned}}$$

or

$$\left(\begin{aligned} x&=\frac{10}{9}-7t\\ y&=\frac{13}{9}+17t\\ z&=9t\end{aligned}\right)$$

Example
Three Planes Intersect in Pairs, Forming Three Parallel Lines



π_1: $x+2y-3z-4=0$ ①
π_2: $5x+y+2z-7=0$ ②
π_3: $7x+5y-4z-18=0$ ③

None of the planes are parallel.

① × 5 − ② $9y-17z-13=0$
① × 7 − ③ $9y-17z-10=0$

Subtract to eliminate y

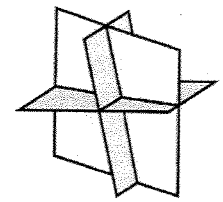
$0z-3=0$

$0z=3$

No solution
∴ three planes do not intersect (and are not parallel)

∴ three planes intersect in pairs along three parallel lines.

Example
Three Planes Intersect at a Single Point



π_1: $3x+2y-z-13=0$ (1)
π_2: $x-y+2z+3=0$ (2)
π_3: $2x+3y-4z-17=0$ (3)

None of the planes are parallel.

(1)×2 + (2) $7x+3y-23=0$ (4)
(1)×4 − (3) $10x+5y-35=0$ (5)

---

(4)×5 − (5)×3 $5x-10=0$

$x=2$

Sub. in (4) $7(2)+3y-23=0$

$y=3$

Sub. in (1) $3(2)+2(3)-z-13=0$

$z=-1$

∴ three planes intersect at the single point $(2,3,-1)$

A Quick Way to Check if Three Planes Intersect at a Single Point

Notice that when three planes intersect at a single point (as shown on the previous page), their normal vectors are not coplanar. In fact, this case is the **only** one in which the three normal vectors are not coplanar.

- Therefore, a quick way to check if three planes intersect at a single point is to see whether or not their normal vectors are coplanar.
  - Recall that if $\vec{u} \cdot \vec{v} \times \vec{w} = 0$, then $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar.
    - If $\vec{u} \cdot \vec{v} \times \vec{w} \neq 0$, then $\vec{u}$, $\vec{v}$ and $\vec{w}$ are **not** coplanar.

Checking if Three Planes Intersect at a Single Point

Suppose three planes have normal vectors $\vec{n_1}$, $\vec{n_2}$ and $\vec{n_3}$.

- If $\vec{n_1} \cdot \vec{n_2} \times \vec{n_3} \neq 0$, then the normal vectors are **not** coplanar and the three planes intersect at a single point.
- If $\vec{n_1} \cdot \vec{n_2} \times \vec{n_3} = 0$, then the normal vectors are coplanar and the three planes do **not** intersect at a single point (they may intersect in another way though).

Example

Determine if the following planes intersect at a single point.

π_1: $2x - y + 3z - 2 = 0$ $\quad \vec{n_1} = (2,-1,3)$

π_2: $x - 3y + 2z + 10 = 0$ $\quad \vec{n_2} = (1,-3,2)$

π_3: $5x - 5y + 8z + 3 = 0$ $\quad \vec{n_3} = (5,-5,8)$

$$\begin{matrix} 1 & -3 & 2 & 1 \\ 5 & -5 & 8 & 5 \end{matrix}$$

$$\begin{aligned} \vec{n_1} \cdot \vec{n_2} \times \vec{n_3} &= (2,-1,3) \cdot (1,-3,2) \times (5,-5,8) \\ &= (2,-1,3) \cdot (-14, 2, 10) \\ &= -28 - 2 + 30 \\ &= 0 \end{aligned}$$

$\therefore$ $\vec{n_1}$, $\vec{n_2}$ and $\vec{n_3}$ are coplanar

$\therefore$ π_1, π_2 and π_3 do not intersect at a single point.