The Product Rule

The Product Rule

If $p(x)=f(x)g(x)$, then $p'(x)=f'(x)g(x)+g'(x)f(x)$.

Proof:

Let $p(x)=f(x)g(x)$.

$$p'(x)=\lim_{h\to 0}\frac{p(x+h)-p(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\left\{g(x+h)\left[\frac{f(x+h)-f(x)}{h}\right]+f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$=\lim_{h\to 0}g(x+h)\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]+\lim_{h\to 0}f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$=g(x)f'(x)+f(x)g'(x)$$

$$=f'(x)g(x)+g'(x)f(x) \quad \blacksquare$$

The Quotient Rule

The Quotient Rule for Derivatives

Let $h(x) = \frac{f(x)}{g(x)}$. If both $f'(x)$ and $g'(x)$ exist, the derivative of $h(x)$ is

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}, \text{ where } g(x) \neq 0.$$

In Leibniz notation, $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - \left[\frac{d}{dx}g(x)\right]f(x)}{[g(x)]^2}$, $g(x) \neq 0$.

Proof:

Let $h(x) = \frac{f(x)}{g(x)}$ — Multiply both sides by $g(x)$.

$g(x)h(x) = f(x)$ — Differentiate both sides with respect to x.

$g'(x)h(x) + h'(x)g(x) = f'(x)$ — Solve for $h'(x)$.

$h'(x) = \frac{f'(x) - g'(x)h(x)}{g(x)}$ — Substitute $h(x) = \frac{f(x)}{g(x)}$.

$= \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$ — Multiply both the numerator and the denominator by $g(x)$.

$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$ ■

The Chain Rule

The Chain Rule

If $w(x)=f\left(g(x)\right)$, then $w'(x)=f'\left(g(x)\right)g'(x)$.

Proof:

Let $w(x)=f\left(g(x)\right)$

$$w'(x)=\lim_{h\to 0}\frac{w(x+h)-w(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(g(x+h)\right)-f\left(g(x)\right)}{h}$$

Assuming that $g(x+h)-g(x)\neq 0$, we can write

$$=\lim_{h\to 0}\left\{\left[\frac{f\left(g(x+h)\right)-f\left(g(x)\right)}{g(x+h)-g(x)}\right]\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$=\lim_{h\to 0}\left[\frac{f\left(g(x+h)\right)-f\left(g(x)\right)}{g(x+h)-g(x)}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

From the previous page:

$$w'(x) = \lim_{h\to 0}\left[\frac{f\left(g(x+h)\right) - f\left(g(x)\right)}{g(x+h)-g(x)}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$\lim_{h\to 0}\left[g(x+h)-g(x)\right]=0,$ let $g(x+h)-g(x)=k$ and $k \to 0$ as $h \to 0$

Therefore,

$$w'(x) = \lim_{k\to 0}\left[\frac{f\left(g(x)+k\right) - f\left(g(x)\right)}{k}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= f'\left(g(x)\right)g'(x) \qquad \blacksquare$$

Note that this proof is not valid for all circumstances, as we made the assumption that $g(x+h)-g(x)\neq 0.$ A proof that covers all cases can be found in advanced calculus textbooks.