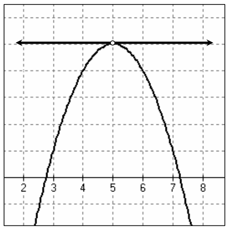
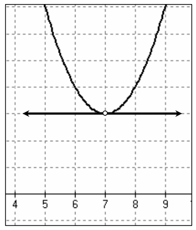
Solving Problems Involving Rates of Change

Maximum and Minimum Values

Let's consider the following sections of two graphs, one of which exhibits a local maximum and the other which exhibits a local minimum:





Question: What is the slope of the tangent where the local maximum/minimum occurs?

ZERO!

We can use the idea that the slope of the tangent where a local maximum or minimum occurs is zero to help us solve problems.

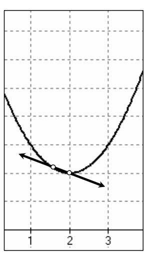
> **<u>Example</u>**
> Show that the minimum value for the function $f(x) = x^2 - 4x + 11$ happens when $x = 2$.

Now, we could just find the vertex of this upward-opening parabola and show that it has an x-value of 2, but let's use our knowledge of slopes of tangents instead, so we are prepared for tougher functions!

If a minimum does indeed occur when $x = 2$, what can we expect the slope of the tangent to be where $x = 2$?

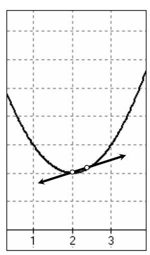
ZERO!

If a minimum occurs when $x = 2$, what can we expect about the slopes of secants through the point at $x = 2$ and points with x-values slightly lower than 2 and slightly higher than 2?



Second x-value slightly less than 2:

> Slope of secant is negative



Second x-values slightly higher than 2:

> Slope of secant is positive

Let's see if these conditions are actually met…

$$f(x) = x^2 - 4x + 11$$

First, let's estimate the slope of the tangent at $x = 2$ with a secant where the second point is slightly to the left of $x = 2$.

$$m_{\text{secant}} = \frac{f(2) - f(1.99)}{2 - 1.99}$$

$$m_{\text{secant}} = \frac{7 - 7.0001}{0.01}$$

$$m_{\text{secant}} = -0.01$$

Now, let's estimate the slope of the tangent at $x = 2$ with a secant where the second point is slightly to the right of $x = 2$.

$$m_{\text{secant}} = \frac{f(2.01) - f(2)}{2.01 - 2}$$

$$m_{\text{secant}} = \frac{7.0001 - 7}{0.01}$$

$$m_{\text{secant}} = 0.01$$

$$m_{\text{secant}} = \frac{f(2) - f(1.99)}{2 - 1.99} = -0.01$$

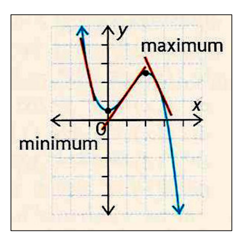
$$m_{\text{secant}} = \frac{f(2.01) - f(2)}{2.01 - 2} = 0.01$$

- In both cases, the approximation yielded results close to zero, one slightly above and the other slightly below.
  - It appears safe to conclude that the slope of the tangent at $x = 2$ is 0, thus implying that a local maximum or minimum occurs where $x = 2$.
- Also, the signs of the secant slopes indicate that the function is decreasing immediately before $x = 2$ and increasing immediately after $x = 2$, which implies that we have a minimum and not a maximum.

> Therefore, we can conclude that the minimum value for $f(x) = x^2 - 4x + 11$ does indeed occur when $x = 2$.

Question for Discussion

How would the solution differ if the previous example dealt with a local maximum instead of a local minimum?



One more example (to be solved on a separate page)…

Verify that a local maximum for the function $f(x) = 3\sin x$ occurs when $x = 450°$.