PROVING TRIGONOMETRIC IDENTITIES

Recall that an identity is a statement of equality that is true for **ALL** values of the variables. For example, the algebraic statement $3x + 2x = 5x$ is an identity, since it is true for all values of x. The equation $3x = 15$, however, is not an identity, as it is only true for $x = 5$.

In grade 11, you proved several trigonometric identities. To do so, you used some simpler identities. Let's do a little review of the identities that we have available in our "toolbox".

First, we have the **reciprocal identities**, with which we have become very familiar:

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

We also have the **quotient identities**:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Don't forget about the **Pythagorean identity**:

$$\sin^2 x + \cos^2 x = 1$$

If we divide the above Pythagorean identity by $\sin^2 x$ or $\cos^2 x$, we arrive at the following two identities, respectively (also called Pythagorean identities):

$$1 + \cot^2 x = \csc^2 x \qquad \tan^2 x + 1 = \sec^2 x$$

Then there are the new identities and formulas that we have recently learned:

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Addition and Subtraction Formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

The key to becoming adept at proving identities is **perseverance**. If you relish the challenge of a good puzzle, this will prove to be a particularly enjoyable section!

Here are a few strategies that are commonly used in proving trigonometric identities:

1) Move from the complex to the simple.
2) Express all functions in terms of sine and cosine.
3) Look for squares and use the Pythagorean identities.
4) Express all functions with the same *argument*.

Check out the following few examples:

Example 1 Prove $1 + \cos x = \dfrac{\sin^2 x}{1 - \cos x}$.

Solution We start with the more complex right side and focus on $\sin^2 x$.

$$\begin{aligned} \frac{\sin^2 x}{1 - \cos x} &= \frac{1 - \cos^2 x}{1 - \cos x} \\ &= \frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x} \\ &= 1 + \cos x \end{aligned}$$

Example 2 Prove $\cos(x + y)\cos(x - y) = \cos^2 x + \cos^2 y - 1$.

Solution The left side contains two familiar factors, so let's start with it. As we move through the proof, we are constantly aware of the terms on the right side that we need to produce on the left side.

$$\begin{aligned} \cos(x + y)\cos(x - y) &= (\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y) \\ &= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y \end{aligned}$$

Noting that $\sin^2 x$ and $\sin^2 y$ do not appear on the right side, we replace them and get

$$\begin{aligned} \cos(x + y)\cos(x - y) &= \cos^2 x \cos^2 y - (1 - \cos^2 x)(1 - \cos^2 y) \\ &= \cos^2 x \cos^2 y - (1 - \cos^2 x - \cos^2 y + \cos^2 x \cos^2 y) \\ &= \cos^2 x \cos^2 y - 1 + \cos^2 x + \cos^2 y - \cos^2 x \cos^2 y \\ &= \cos^2 x + \cos^2 y - 1 \end{aligned}$$

Example 3 Prove $\dfrac{\sin 2x}{1 - \cos 2x} = 2 \csc 2x - \tan x$.

Solution We express csc and tan in terms of sin and cos. Next we note the presence of two different arguments, $2x$ and x. We work with each side separately to achieve a common result.

$$\begin{aligned} \frac{\sin 2x}{1 - \cos 2x} &= \frac{2 \sin x \cos x}{1 - (1 - 2\sin^2 x)} \\ &= \frac{2 \sin x \cos x}{2 \sin^2 x} \\ &= \frac{\cos x}{\sin x} \end{aligned}$$

$$\begin{aligned} 2 \csc 2x - \tan x &= \frac{2}{\sin 2x} - \tan x \\ &= \frac{2}{2 \sin x \cos x} - \frac{\sin x}{\cos x} \\ &= \frac{2 - 2\sin^2 x}{2 \sin x \cos x} \\ &= \frac{1 - \sin^2 x}{\sin x \cos x} \\ &= \frac{\cos^2 x}{\sin x \cos x} \\ &= \frac{\cos x}{\sin x} \end{aligned}$$

Therefore $\dfrac{\sin 2x}{1 - \cos 2x} = 2 \csc 2x - \tan x$