Properties of Definite Integrals

and

The Mean Value Theorem

Properties of Definite Integrals

Question

How would $\int_a^b f(x)dx$ compare to $\int_b^a f(x)dx$?

With $\int_b^a f(x)dx$, we are integrating in the opposite direction than with $\int_a^b f(x)dx$.

That is, if we moved from left to right over the interval [*a*, *b*], we would be moving from right to left over the interval [*b*, *a*].

Therefore, the values of Δ*x* in the corresponding Riemann sums would be opposite, and thus the terms of each Riemann sum would have the opposite signs.

Recall:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

The above suggests that $\int_b^a f(x)dx$ and $\int_a^b f(x)dx$ have the same value but opposite sign.

That is,

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Note:
This rule is an extension of the definition of the definite integral.

Rules for Definite Integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition
2. *Zero:* $\int_a^a f(x)\,dx = 0$ — Also a definition
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

   $$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$
7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ — $g = 0$

Although proofs of the these rules are beyond the scope of this course, a brief discussion should be enough to convince us that they are valid.

Examples

Complete each of the following on a separate page.

1) Suppose

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Find each of the following integrals, if possible.

(a) $\int_4^1 f(x)\,dx$ **(b)** $\int_{-1}^4 f(x)\,dx$ **(c)** $\int_{-1}^1 [2f(x) + 3h(x)]\,dx$

(d) $\int_0^1 f(x)\,dx$ **(e)** $\int_{-2}^2 h(x)\,dx$ **(f)** $\int_{-1}^4 [f(x) + h(x)]\,dx$

2) Show that the value of $\int_0^1 \sqrt{1 + \cos x}\,dx$ is less than 3/2.

Average Value of a Function

We know that the *average* of *n* numbers is simply the sum of the numbers divided by *n*.

How could we define the average value of a function *f* over the closed interval [*a*, *b*], since there are infinitely many values?

Consider the following approach:

1) Break up the interval [*a*, *b*] into a large number of regular subintervals.
2) If there are *n* regular subintervals, the length of each interval would be

$$\Delta x = \frac{b-a}{n}$$

3) Take some number c_k from each of the *n* subintervals.
   - The average of the corresponding *y*-values is

$$\frac{f(c_1)+f(c_2)+\ldots+f(c_n)}{n}$$

4) $$\frac{f(c_1)+f(c_2)+\ldots+f(c_n)}{n} = \frac{1}{n}\left[f(c_1)+f(c_2)+\ldots+f(c_n)\right]$$

$$= \frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

$$= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k)$$

$$= \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Notice that this expression is $\frac{1}{b-a}$ multiplied by a Riemann sum for *f* on [*a*, *b*].

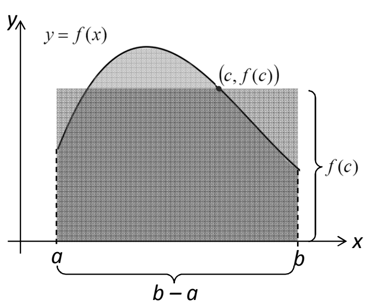
So, as *n* approaches infinity, the average value has a limit!

DEFINITION Average (Mean) Value

If *f* is integrable on [*a*, *b*], its **average (mean) value** on [*a*, *b*] is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Mean Value Theorem for Definite Integrals



- Consider the function *f*(*x*) on the interval [*a*, *b*], as shown on the left.
- We know that the area under the curve is $\int_a^b f(x)dx.$
- Now, consider a rectangle with base *b* – *a* and a height given by a point (*c*, *f*(*c*)) on the curve.
- The area of this rectangle is *f*(*c*)(*b* – *a*).
- If (*c*, *f*(*c*)) is the minimum of *f*, the area of the rectangle is smaller than $\int_a^b f(x)dx.$
- If (*c*, *f*(*c*)) is the maximum of *f*, the area of the rectangle is larger than $\int_a^b f(x)dx.$
- Therefore, there must be some point(s) on the curve, higher than the minimum and lower than the maximum, such that the area of the rectangle is the same as $\int_a^b f(x)dx.$

- That is, there must be some (*c*, *f*(*c*)) such that $f(c)(b-a) = \int_a^b f(x)dx$

  and thus $f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)dx.}_{\text{average value of } f \text{ on } [a,\, b]}$

- The above result indicates that at some point in the interval [*a*, *b*], the value of the function is equal to the function's average value over the interval.
- The idea is known as the Mean Value Theorem for Definite Integrals.

THEOREM 3 The Mean Value Theorem for Definite Integrals

If *f* is continuous on [*a*, *b*], then at some point *c* in [*a*, *b*],

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

<u>Example</u> (complete on a separate page)

With the aid of a calculator, find the average value of *f*(*x*) = 4 – x^2 on the interval [0, 3]. At what point(s) in the interval does the function assume its average value?