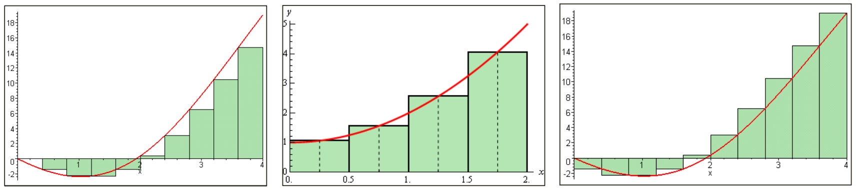
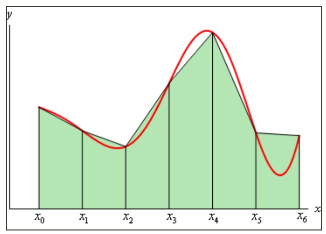
The Trapezoidal Rule

We have seen that LRAM, MRAM and RRAM can be used to approximate integrals.



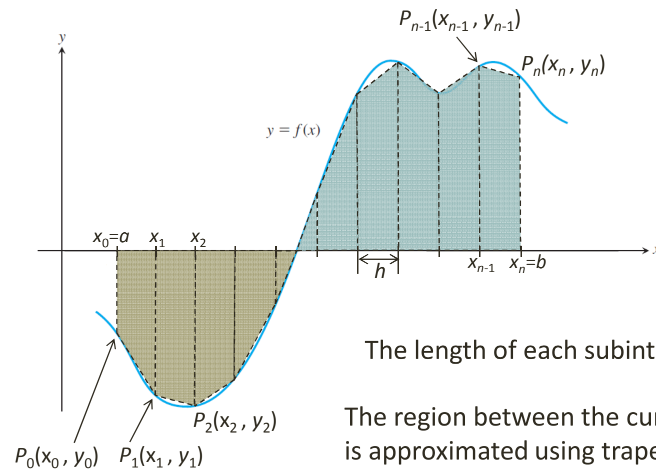
Shapes other than rectangles can also be used to approximate integrals, often with more efficiency (they give better approximations than rectangles, especially for a small number of subintervals).

An example is trapezoidal approximation:



Trapezoidal Approximation

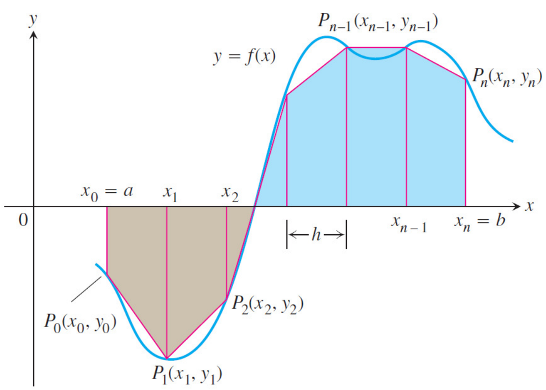
Consider the following graph of $y = f(x)$, in which the interval $[a, b]$ is partitioned into n subintervals of equal length.



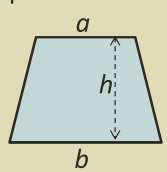
The length of each subinterval is $h = \dfrac{b-a}{n}$

The region between the curve and the x-axis is approximated using trapezoids.

Question for Discussion:
What happens with the fifth "trapezoid" in the diagram above?



Recall the area of a trapezoid:



$$A = h\left(\frac{a+b}{2}\right)$$

Therefore,

$$\int_a^b f(x)dx \approx h\left(\frac{y_0+y_1}{2}\right)+h\left(\frac{y_1+y_2}{2}\right)+\ldots+h\left(\frac{y_{n-1}+y_n}{2}\right)$$

$$= h\left(\frac{y_0}{2}+y_1+y_2+\ldots+y_{n-1}+\frac{y_n}{2}\right)$$

$$= \frac{h}{2}(y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n)$$

$$= \frac{h}{2}(y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n)$$ *(from the previous slide)*

Question for Discussion:
Why is the above expression equal to the average of the corresponding LRAM and RRAM?

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into n subintervals of equal length $h = (b - a)/n$.
Equivalently,

$$T = \frac{\text{LRAM}_n + \text{RRAM}_n}{2},$$

where LRAM_n and RRAM_n are the Riemann sums using the left and right endpoints, respectively, for f for the partition.

Examples

Complete the following on a separate page.

1) a) Use the Trapezoidal Rule with n = 4 to approximate $\int_1^2 x^2 dx.$

b) Compare your approximation to the exact value. Express the error as a percentage.

> *Note: The increased efficiency of the Trapezoidal Rule over rectangular approximations makes it a faster algorithm for machine computation.*

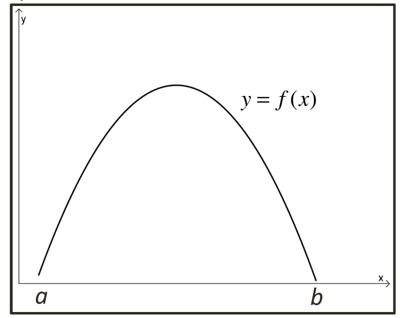
2) An observer measures the outside temperature every hour from noon until midnight, as shown in the table below.

Time	N	1	2	3	4	5	6	7	8	9	10	11	M
Temp	63	65	66	68	70	69	68	68	65	64	62	58	55

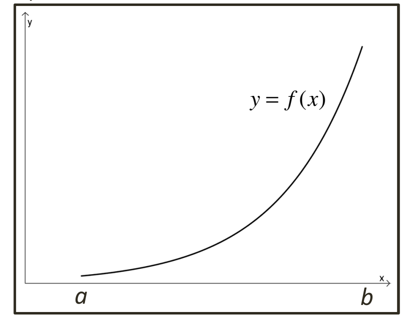
Estimate the average temperature for the 12-hour period.

3) Consider the two graphs shown below.

i)



ii)

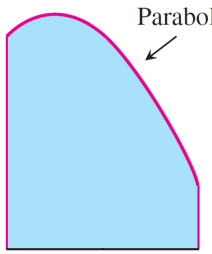


a) For each graph, state whether the Trapezoidal Rule will underestimate or overestimate $\int_a^b f(x)dx.$

b) In general, when does the Trapezoidal Rule underestimate the integral and when does it overestimate the integral?

Simpson's Rule

Notice that the shortcoming of the Trapezoidal Rule is that it approximates curved arcs with straight segments.



Simpson's Rule uses curved pieces to approximate these curved arcs.

- Specifically, Simpson's Rule uses *parabolic arcs* in the approximation.

Simpson's Rule

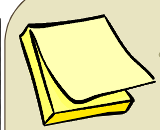
To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into an *even* number n of subintervals of equal length $h = (b - a)/n$.

Use your textbook or other resources to investigate the derivation of Simpson's Rule.

Thomas Simpson
1720 - 1761



NOTES:

- Due to the use of parabolic arcs in the approximation, Simpson's Rule is even more efficient than the Trapezoidal Rule, and therefore faster for computer approximations of integrals.
- The Trapezoidal Rule was discovered long before Thomas Simpson. He never laid claim to it, but rather only mentioned it in one of his textbooks.

Example

Complete the following on a separate page.

a) Use Simpson's Rule with n = 4 to approximate $\int_0^2 5x^4 dx.$

b) Compare your approximation to the exact value. Express the error as a percentage.