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NAME: SOLUTIONS

KNOW /12	APP /12	INQ /12	COMM /6

MCV4U1 - UNIT 5 – DERIVATIVES OF EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS
TEST

ROUND ALL ANSWERS TO THE NEAREST HUNDREDTH UNLESS OTHERWISE STATED

1) Differentiate each of the following and factor/simplify your answer as much as possible. (*K – 2, 2, 3, 3 marks*)

a) $f(x) = (\sin x^2 + \cos x)^3$

$f'(x) = 3(\sin x^2 + \cos x)^2(2x\cos x^2 - \sin x)$

b) $f(x) = \cos(2^{3x})$

$f'(x) = (-\sin 2^{3x})2^{3x}(\ln 2)(3)$
$= -3(\ln 2)(2^{3x})\sin 2^{3x}$

c) $f(x) = e^{4x}\sin^2 x$

$f'(x) = e^{4x}(4)\sin^2 x + 2(\sin x)(\cos x)e^{4x}$
$= 2e^{4x}\sin x(2\sin x + \cos x)$

d) $f(x) = \dfrac{3\tan x}{e^{x^2}}$

$f'(x) = \dfrac{3(\sec^2 x)e^{x^2} - e^{x^2}(2x)(3\tan x)}{(e^{x^2})^2}$

$= \dfrac{3e^{x^2}(\sec^2 x - 2x\tan x)}{(e^{x^2})^2}$

$= \dfrac{3(\sec^2 x - 2x\tan x)}{e^{x^2}}$

2) Determine, to the nearest hundredth, the slope of the tangent to $y = 5^{\sin 8x}$ where $x = 3$. (*K – 2 marks*)

$y' = 5^{\sin 8x}(\ln 5)(\cos 8x)(8)$

At $x = 3$,
$y' = 5^{\sin 8(3)}(\ln 5)(\cos 8(3))(8)$
$\doteq 1.27$

$\therefore$ slope is approximately 1.27

3) Determine the intervals of increase and decrease for the function $f(x) = 2\cos x + \sqrt{3}x$ on the domain $0 \le x \le 2\pi$. (*A – 6 marks*)

$f'(x) = -2\sin x + \sqrt{3}$

For critical #'s

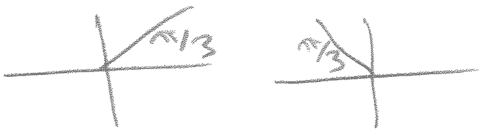
$0 = -2\sin x + \sqrt{3}$

$2\sin x = \sqrt{3}$

$\sin x = \frac{\sqrt{3}}{2}$

R.A.A $= \frac{\pi}{3}$

Quadrants I + II



$x = \frac{\pi}{3}$ or $x = \frac{2\pi}{3}$

Interval	$0 \le x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{2\pi}{3}$	$\frac{2\pi}{3} < x \le 2\pi$
Sign of $f'(x)$	+	−	+
	↗	↘	↗

∴ - increasing on $[0, \frac{\pi}{3})$ and $(\frac{2\pi}{3}, 2\pi]$

- decreasing on $(\frac{\pi}{3}, \frac{2\pi}{3})$

4) Determine, to the nearest hundredth, the coordinates of all local maximum and local minimum points for the function $f(x) = \frac{e^x}{x^3}$. For each point, be sure to state whether it is a maximum or minimum that occurs. (*A - 6 marks*)

$f'(x) = \frac{e^x x^3 - 3x^2 e^x}{x^6}$

$= \frac{e^x x^2 (x-3)}{x^6}$

$= \frac{e^x (x-3)}{x^4}$

Critical #'s are 0, 3

Interval	$x < 0$	$0 < x < 3$	$x > 3$
Sign of $f'(x)$	−	−	+
	↘	↘	↗

∴ local minimum at

(3, 0.74)

5) A certain population of squirrels is represented by the function $P(t) = 3te^{-\frac{t}{3}}$, where P is the number of squirrels, **in hundreds**, after t weeks.

a) After how many weeks is the number of squirrels a maximum? (Be sure to show your check for a maximum). (*I – 5 marks*)

$P'(t) = 3e^{-\frac{t}{3}} + e^{-\frac{t}{3}}\left(-\frac{1}{3}\right)(3t)$

$= 3e^{-\frac{t}{3}} - te^{-\frac{t}{3}}$

$= e^{-\frac{t}{3}}(3-t)$

For max,

$0 = e^{-\frac{t}{3}}(3-t)$

$t = 3$

Check max:

Interval	$t<3$	$t>3$
Sign of $P'(t)$	$+$ ↗	$-$ ↘

$\therefore$ maximum number of squirrels at 3 weeks

b) Determine the maximum number of squirrels in this population. (*I – 1 mark*)

$P(3) = 3(3)e^{-\frac{3}{3}}$

$\doteq 3.31$

$\therefore$ 331 squirrels

6) Determine the equation of the tangent to the curve $y = \tan 2x - 4\tan x$ at the point where $x = \pi$. Give your answer in exact form. (*I – 6 marks*)

$y' = 2\sec^2 2x - 4\sec^2 x$

At $x = \pi$,

$y' = 2\sec^2(2\pi) - 4\sec^2\pi$

$= \frac{2}{\cos^2(2\pi)} - \frac{4}{\cos^2\pi}$

$= 2 - 4$

$= -2$

When $x = \pi$, $y = 0$

$\therefore y = mx + b$

$y = -2x + b$

$0 = -2\pi + b$

$b = 2\pi$

$\therefore y = -2x + 2\pi$

7) Pauline claims that the graph of $y = \dfrac{3^{x-2}}{e^{x+4}}$ will have at least one local maximum or local minimum value. Is Pauline's claim correct? (*C – 3 marks*)

$$y' = \frac{3^{x-2}\ln 3\, e^{x+4} - e^{x+4}3^{x-2}}{(e^{x+4})^2}$$

$$= \frac{3^{x-2}e^{x+4}(\ln 3 - 1)}{(e^{x+4})^2}$$

$$= \frac{3^{x-2}(\ln 3 - 1)}{e^{x+4}}$$

Pauline is incorrect.

3^{x-2} and e^{x+4} are always positive, since the bases are positive. $\ln 3 - 1$ is also positive (approx. 0.1)

$\therefore$ y' is always positive

$\therefore$ y always increasing

$\therefore$ no local max/min

8) Prove that the derivative of $y = \csc x$ is $y' = -\cot x \csc x$. (*C – 3 marks*)

$$y = \csc x$$

$$= \frac{1}{\sin x}$$

$$= (\sin x)^{-1}$$

$$y' = -(\sin x)^{-2}\cos x$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right) \rightarrow = -\cot x \csc x$$

<u>CHALLENGE QUESTION</u> – COMPLETE ONLY IF TIME PERMITS

Determine y' if $x^y = y^x$.

$$x^y = y^x$$

$$\ln x^y = \ln y^x$$

$$y \ln x = x \ln y$$

$$y'\ln x + \frac{1}{x}y = (1)\ln y + \frac{1}{y}y'x$$

$$y'\ln x - y'\frac{x}{y} = \ln y - \frac{y}{x}$$

$$\rightarrow y'\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

or

$$y' = \frac{xy\ln y - y^2}{xy\ln x - x^2}$$