The Definite Integral

Estimating with Finite Sums

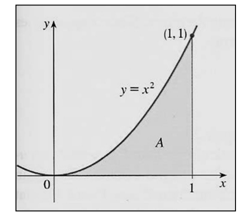
The Two Types of Calculus

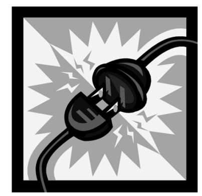
Up to this point, the main focus of our calculus studies has been instantaneous rates of change (tangent slopes).

- This type of calculus is called **differential calculus.**

Another branch of calculus addresses the problem of finding areas under curves.

- This type of calculus is called **integral calculus.**





It may seem that these two types of calculus are unrelated, but as we will soon see, they are indeed connected!

Setting the Stage

Consider the following problem:

A train moves along a track at a steady rate of 125 km/h. If the train travels for 4 hours, what is the total distance traveled?

Solution:

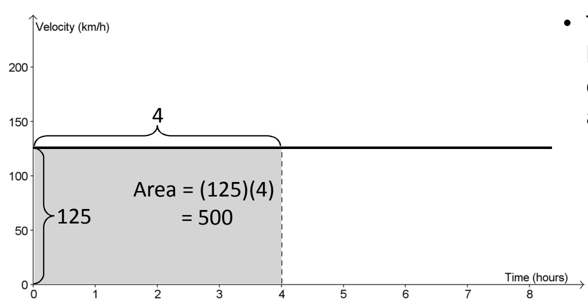
distance = *velocity* × *time*

= (125)(4)

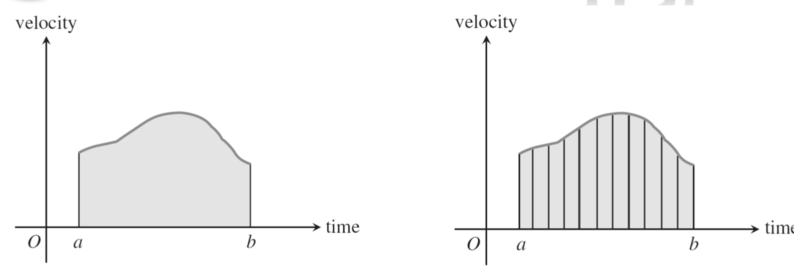
= 500 km

Notice that the distance traveled is actually the area under the graph shown below.

- This connection could be made for any constant velocity and any length of time.



If the velocity was not constant, would the area under the graph still give the distance traveled? **YES!**



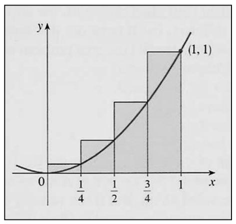
- To understand this idea, consider partitioning the time interval [*a*, *b*] into many small subintervals, as shown above.
- **If these subintervals are made small enough**, the velocity over each subinterval would essentially be constant and the narrow strips depicted above would closely resemble rectangles.
- Just as the total area can be found be summing the areas of the nearly rectangular strips, the total distance can be found by summing the small distances traveled over the tiny time subintervals.

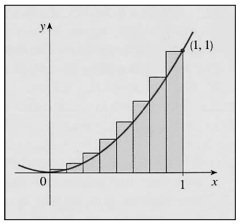
Rectangular Approximation Method (RAM)

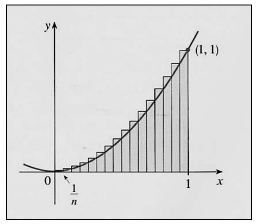
As suggested in the previous slide, we will be using rectangles to approximate the area under graphs.

Some important points:

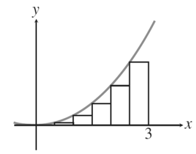
1) The more rectangles (subintervals) we use, the better the approximation of the area under the curve.



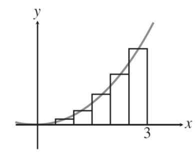




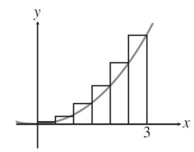
2) We can choose different points at which the rectangle touches the curve.



LRAM
The height of each rectangle is determined by evaluating the function at the left-hand endpoint of the subinterval.



MRAM
The height of each rectangle is determined by evaluating the function at the midpoint of the subinterval.



RRAM
The height of each rectangle is determined by evaluating the function at the right-hand endpoint of the subinterval.

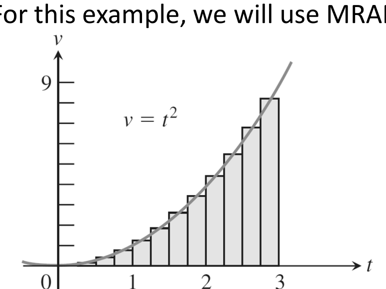
Question for Discussion:
Will LRAM always give an underestimate and will RRAM always give an overestimate of the area under a curve?

An Example

A particle moves along a straight path with velocity $v(t)=t^2$, where $t \geq 0$. If v is measured in m/s, estimate the particle's distance traveled in the first three seconds.

Solution

For this example, we will use MRAM with subintervals of length Δt = ¼.



- The base of each rectangle is ¼.
- The height of each rectangle is found by evaluating $v(t)$ at the midpoint of the subinterval.

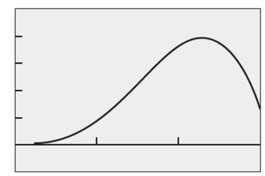
The sum of the rectangular areas is:

$$\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)^2+\left(\frac{1}{4}\right)\left(\frac{3}{8}\right)^2+\left(\frac{1}{4}\right)\left(\frac{5}{8}\right)^2+\left(\frac{1}{4}\right)\left(\frac{7}{8}\right)^2+.....+\left(\frac{1}{4}\right)\left(\frac{21}{8}\right)^2+\left(\frac{1}{4}\right)\left(\frac{23}{8}\right)^2\approx 8.98$$

Therefore, the particle moves approximately 8.98 metres in the first three seconds.

Examples

Solve each of the following on a separate page.



1) The graph of $f(x) = x^2\sin x$ on the interval [0, 3] is shown above.
   a) Use LRAM with 9 subintervals to estimate the area under the curve from x=0 to x=3.
   b) Use a calculator program to estimate the area with 1000 subintervals.

2) A particle moves along a straight path with velocity $v(t) = 2t^3$, where $t \geq 0$. If v is measured in m/s, use RRAM with 10 subintervals to estimate the particle's distance traveled from t=2 to t=4.

3) Use a calculator program to estimate the volume of a sphere with radius 4.
   Hint: A sphere can be generated by revolving a circle about the x-axis. Cylinders can be used to approximate slices of the sphere.