# Double Angle Formulas

In this section we use the addition formulas for sine, cosine, and tangent to generate some frequently used trigonometric relationships.
If we start with

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

then, setting $a = b = x$ gives

$$\sin(x + x) = \sin x \cos x + \cos x \sin x$$
$$\sin 2x = 2 \sin x \cos x$$

**Double Angle Formula for Sine**

$$\sin 2x = 2 \sin x \cos x$$

**Example 1** If $\sin x = \frac{4}{5}, \frac{\pi}{2} < x < \pi$, find the value of $\sin 2x$.

**Solution** Now $\sin 2x = 2 \sin x \cos x$

and, since $\frac{\pi}{2} < x < \pi$,

$$\cos x = -\tfrac{3}{5}$$

Therefore $\sin 2x = 2\left(\tfrac{4}{5}\right)\left(-\tfrac{3}{5}\right) = -\tfrac{24}{25}$

A similar development produces the Double Angle Formula for Cosine.
If we start with

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

then, letting $a = b = x$ gives

$$\cos(x + x) = \cos x \cos x - \sin x \sin x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

The identity $\sin^2 \theta + \cos^2 \theta = 1$ is applied to the original result to produce two alternative forms of the formula.

$$\cos 2x = \cos^2 x - (1 - \cos^2 x) \qquad \cos 2x = (1 - \sin^2 x) - \sin^2 x$$
$$= 2 \cos^2 x - 1 \qquad\qquad = 1 - 2 \sin^2 x$$

**Double Angle Formulas for Cosine**

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 2 \cos^2 x - 1$$
$$\cos 2x = 1 - 2 \sin^2 x$$

The three results are equivalent, but as you gain experience working with these formulas, you will learn that one form may be superior to the others in a particular problem.

**Example 2** If $\cos a = \frac{2}{3}$, find the value of $\cos 4a$.

**Solution** We must express $\cos 4a$ in terms of trigonometric functions of $a$. This is accomplished by applying the Double Angle Formula for Cosine twice.

$$\begin{aligned}
\cos 4a &= \cos[2(2a)] \\
&= 2\cos^2 2a - 1 \\
&= 2(2\cos^2 a - 1)^2 - 1 \\
&= 2[2\left(\tfrac{2}{3}\right)^2 - 1]^2 - 1 \\
&= 2\left(-\tfrac{1}{9}\right)^2 - 1 \\
&= -\tfrac{79}{81}
\end{aligned}$$

The application of the Double Angle Formula for Cosine in the next example should be examined carefully.

**Example 3** Evaluate $\sin \frac{\pi}{8}$.

**Solution** Since

$$\cos \frac{\pi}{4} = \cos 2\left(\frac{\pi}{8}\right)$$

the Double Angle Formula gives

$$\cos \frac{\pi}{4} = 1 - 2\sin^2 \frac{\pi}{8}$$

Solve for $\sin \frac{\pi}{8}$:

$$\sin^2 \frac{\pi}{8} = \frac{1 - \cos \frac{\pi}{4}}{2}$$

Since $\sin \frac{\pi}{8}$ is positive,

$$\begin{aligned}
\sin \frac{\pi}{8} &= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}} \\
&= \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}
\end{aligned}$$

Now we develop the Double Angle Formula for Tangent.
If we start with

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

then, setting $a = b = x$ gives

$$\tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x}$$

Therefore

**Double Angle Formula for Tangent**

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

**Example 4** If $\tan x = \frac{4}{3}$, $\pi < x < 2\pi$, find the value of $\tan \frac{x}{2}$.

**Solution**

$$\tan x = \frac{4}{3}$$

$$\tan\left[2\left(\frac{x}{2}\right)\right] = \frac{4}{3}$$

$$\frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \frac{4}{3}$$

$$6 \tan \frac{x}{2} = 4 - 4 \tan^2 \frac{x}{2}$$

$$2 \tan^2 \frac{x}{2} + 3 \tan \frac{x}{2} - 2 = 0$$

$$\left(2 \tan \frac{x}{2} - 1\right)\left(\tan \frac{x}{2} + 2\right) = 0$$

Therefore

$$\tan \frac{x}{2} = \frac{1}{2} \quad \text{or} \quad \tan \frac{x}{2} = -2$$

Since $\pi < x < 2\pi$, we have $\frac{\pi}{2} < \frac{x}{2} < \pi$, so

$$\tan \frac{x}{2} = -2$$