The Product Rule

The Product Rule

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + g'(x)f(x)$.

Proof:

Let $p(x) = f(x)g(x)$.

$$\begin{aligned}
p'(x) &= \lim_{h\to 0}\frac{p(x+h)-p(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left\{g(x+h)\left[\frac{f(x+h)-f(x)}{h}\right]+f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\} \\
&= \lim_{h\to 0} g(x+h)\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]+\lim_{h\to 0} f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] \\
&= g(x)f'(x)+f(x)g'(x) \\
&= f'(x)g(x)+g'(x)f(x) \qquad \blacksquare
\end{aligned}$$

The Quotient Rule

The Quotient Rule for Derivatives

Let $h(x) = \dfrac{f(x)}{g(x)}$. If both $f'(x)$ and $g'(x)$ exist, the derivative of $h(x)$ is

$h'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$, where $g(x) \neq 0$.

In Leibniz notation, $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\left[\frac{d}{dx}f(x)\right]g(x) - \left[\frac{d}{dx}g(x)\right]f(x)}{[g(x)]^2}$, $g(x) \neq 0$.

Proof:

Let $h(x) = \dfrac{f(x)}{g(x)}$ — Multiply both sides by $g(x)$.

$g(x)h(x) = f(x)$ — Differentiate both sides with respect to x.

$g'(x)h(x) + h'(x)g(x) = f'(x)$ — Solve for $h'(x)$.

$h'(x) = \dfrac{f'(x) - g'(x)h(x)}{g(x)}$ — Substitute $h(x) = \dfrac{f(x)}{g(x)}$.

$= \dfrac{f'(x) - g'(x)\dfrac{f(x)}{g(x)}}{g(x)}$ — Multiply both the numerator and the denominator by $g(x)$.

$h'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$ ■