We know

$$\sin\frac{\pi}{6} = \frac{1}{2} \quad \text{and} \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\therefore \sin\frac{\pi}{6} + \sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}$$

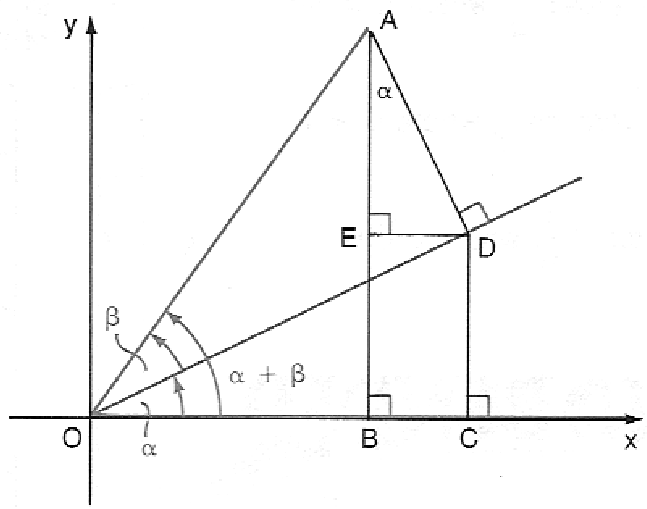
But

$$\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) = \sin\frac{\pi}{2} = 1$$

$$\therefore \sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right) \neq \sin\frac{\pi}{6} + \sin\frac{\pi}{3}$$

In this section we will derive formulas for the sine and cosine of the sum and difference of two angles.

To derive formulas for functions of $\alpha + \beta$, we place the angles α and β with reference to coordinate axes as shown. Taking A on the terminal arm of $(\alpha + \beta)$ we draw the following perpendiculars — AD perpendicular to the terminal arm of α, $AB \perp OX$, $DC \perp OX$, and $DE \perp AB$.



$$OB = OC - BC$$

But

$$\frac{OB}{OA} = \cos(\alpha + \beta) \Rightarrow OB = OA\cos(\alpha + \beta) \qquad ①$$

$$\frac{OC}{OD} = \cos\alpha \qquad \Rightarrow OC = OD\cos\alpha \qquad ②$$

In $\triangle AED$, $\angle EAD = \alpha$,

$$\therefore \frac{ED}{AD} = \sin\alpha \Rightarrow ED = AD\sin\alpha$$

Since $\quad BC = ED, \quad BC = AD\sin\alpha \qquad ③$

Substitute ①, ②, and ③ into OB = OC − BC,

$$OA \cos(\alpha + \beta) = OD \cos \alpha - AD \sin \alpha$$

Divide by OA.

$$\cos(\alpha + \beta) = \frac{OD}{OA} \cos \alpha - \frac{AD}{OA} \sin \alpha$$

But $\frac{OD}{OA} = \cos \beta$ and $\frac{AD}{OA} = \sin \beta$

Consequently,

$$\boxed{\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

$\cos(-\theta) = \cos \theta$

$\sin(-\theta) = -\sin \theta$

Replacing β with $-\beta$, we have

$$\cos(\alpha + (-\beta)) = \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta)$$

$$\boxed{\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta}$$

To develop formulas for $\sin(\alpha - \beta)$ and $\sin(\alpha + \beta)$ we replace α with $90° - \alpha$ and β with $-\beta$. Then

$$\cos((90° - \alpha) - \beta) = \cos(90° - \alpha) \cos \beta + \sin(90° - \alpha) \sin \beta$$

$$\cos(90° - (\alpha + \beta)) = \cos(90° - \alpha) \cos \beta + \sin(90° - \alpha) \sin \beta$$

Using $\cos(90° - \theta) = \sin \theta$ and $\sin(90° - \theta) = \cos \theta$ above, we get

$$\boxed{\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta}$$

Replacing β with $-\beta$, we get

$$\sin(\alpha - \beta) = \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta)$$

or

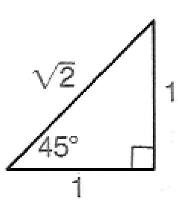
$$\boxed{\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta}$$

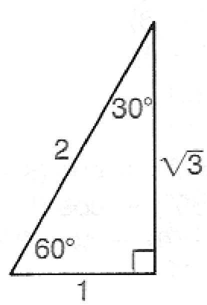
These formulas are called the addition and subtraction formulas.

EXAMPLE 1. Find sin 75° without using tables.

SOLUTION:

Let $\alpha = 45°$, $\beta = 30°$, and $\alpha + \beta = 75°$.





$$\begin{aligned}
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin 75° &= \sin(45° + 30°) \\
&= \sin 45° \cos 30° + \cos 45° \sin 30° \\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3} + 1}{2}\right) \\
&= \tfrac{1}{4}(\sqrt{6} + \sqrt{2})
\end{aligned}$$

$E=mc^2$

EXAMPLE 2. Find a formula for $\tan(\alpha + \beta)$.

SOLUTION:

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Divide the numerator and denominator by $\cos\alpha\cos\beta$.

$$\tan(\alpha + \beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

which reduces to

$$\boxed{\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

Replacing β with $(-\beta)$, we get

$\tan(-\theta) = -\tan\theta$

$$\boxed{\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}}$$

We can also use the addition formulas to find the functions of twice an angle as in the following example.

EXAMPLE 3. Develop a formula for $\cos 2\alpha$.

SOLUTION:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Replacing β with α, we have

$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$

$$\boxed{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$$