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INTRODUCTION

It would be difficult to dispute that the Riemann hypothesis is currently the most famous problem in mathematics. Although the hypothesis itself is certainly not common knowledge among the general public, its reach extends far beyond the discipline of pure mathematics and can be shown to have an impact on the daily activities of even those most removed from mathematical academia. Carrying a reward of one million dollars for the first person to prove or disprove it, the hypothesis is one of the seven Millennium Problems set by the Clay Mathematical Institute of Camebridge, Massachusetts in 2000. To date, the only one of these problems that has been cracked is the Poincaré Conjecture, which was solved by Grigori Perelman in 2003. Before considering the significance of the Riemann hypothesis in the field of mathematics and other disciplines, let us first examine its assertion and development.

INTRODUCING THE RIEMANN ZETA FUNCTION

Reminiscent of one of the other most famous and challenging problems in mathematical history, Fermat's Last Theorem, the Riemann hypothesis is relatively simple to state and comprehend. Unlike Fermat's Last Theorem, however, the Riemann hypothesis has yet to be proven. Posed by German mathematician Bernhard Riemann in 1859, the hypothesis is centred about the following function, called the Riemann zeta function (ζ is the Greek letter zeta):

$$\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+..... \quad =\sum_{k=1}^{\infty}\frac{1}{k^s}$$

The use of sigma notation allows the function to be written more conveniently as

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$$

The zeta function can be evaluated at chosen values of *s* by using techniques for summing infinite series. For many values of *s*, the function is convergent. That is, for certain values of *s*, as more and more terms are added in the series, the result approaches a specific value. In these cases, the infinite sum is equal to the limiting value. A famous case of convergence with this infinite series occurs when we choose $s=2$. Known as the Basel problem (named after the Swiss city Basel), Leonhard Euler determined the following result in 1735:

$$\zeta(2)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+..... \quad =\frac{\pi^2}{6}$$

Similarly, the series converges when $s = 3$:

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots.. \quad \approx 1.202056903159594$$

There are also values of s for which the infinite series of the Riemann zeta function does not approach a finite value. In these cases, the series is said to be divergent. A simple example occurs when we choose $s = -1$:

$$\zeta(-1) = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}}$$

$$= 1 + 2 + 3 + 4 + \ldots$$

It is obvious that the sum of the natural numbers shown above does not have a limit, but rather grows larger and larger as more terms are added. Interestingly, however, Riemann explained how it is possible to assign a finite value to this sum and others like it in the context of the zeta function. Riemann also stated that not only integers should be considered for s in the zeta function, but rather all real numbers. He proved that if s is any real number greater than 1, then the zeta function converges. Riemann did not stop with real numbers, however. He also considered the use of complex numbers in the zeta function.

Complex Numbers

The complex numbers are a set of numbers that encompass not only all real numbers, but also imaginary numbers. An imaginary number results from the square root of a negative number. To denote imaginary numbers, mathematicians use i, which is defined as

$$i = \sqrt{-1}$$

Using the number i, we can apply familiar techniques for simplifying radicals, as shown below.

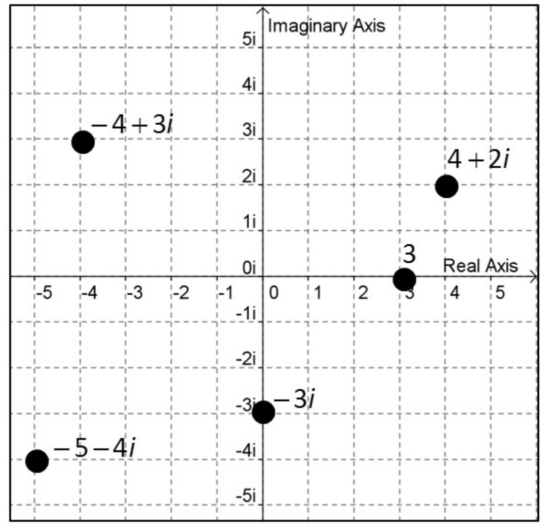
$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$$

$$\sqrt{-18} = \sqrt{18}\sqrt{-1} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3\sqrt{2}i$$

Complex numbers consist of a real part and an imaginary part. The following are all examples of complex numbers:

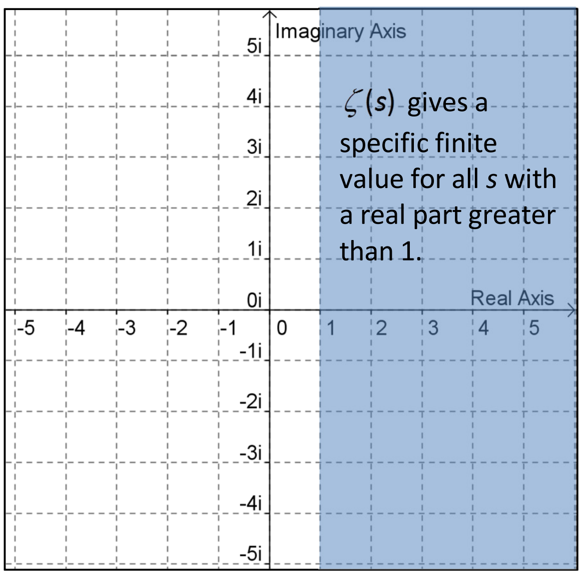
$3+2i$	(real part is 3, imaginary part is 2)
$7-\sqrt{3}i$	(real part is 7, imaginary part is $-\sqrt{3}$)
-9	(real part is -9, imaginary part is 0)
$4i$	(real part is 0, imaginary part is 4)

Since imaginary numbers do not exist on the real number line, complex numbers are plotted on the complex plane, in which the horizontal axis represents the real part and the vertical axis represents the imaginary part. The diagram on the right illustrates the complex plane, along with some plotted points:

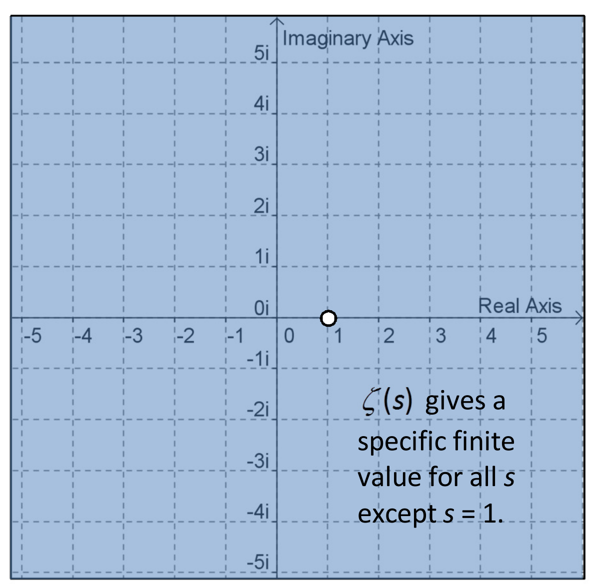


THE RIEMANN ZETA FUNCTION WITH COMPLEX NUMBERS

Not only did Riemann prove that the zeta function converges for any real number greater than 1, he also showed that if *s* is any complex number to the right of the vertical line through 1, the zeta function converges. In other words, the function will approach a finite value for all complex numbers with a real part greater than 1, as shown in the diagram on the left below. For example, $s = 2+3i$, $s = 6.5$ and $s = 1.7-6i$ will cause the zeta function to converge. Note that the use of complex numbers in the zeta function will cause it to converge to a complex number. Using methods from a branch of mathematics called complex analysis, specifically a technique known as analytic continuation, Riemann showed that the domain for which the zeta function converges can further be extended to include all complex numbers, with one exclusion. The exception is the case where *s* = 1, in which the zeta function will not give a finite result. In complex analysis, the Riemann zeta function fits the description of a holomorphic function, giving it some special properties that allow analytic continuation to be applied.

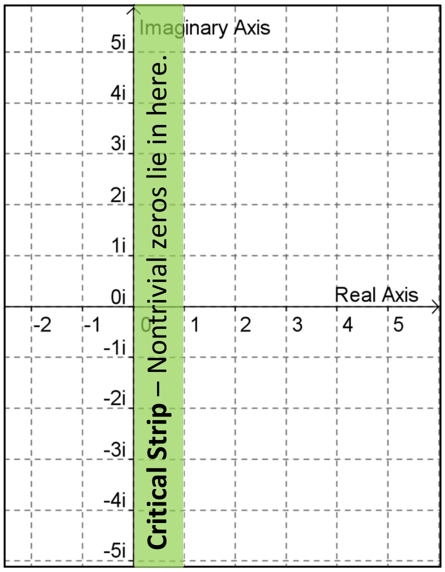


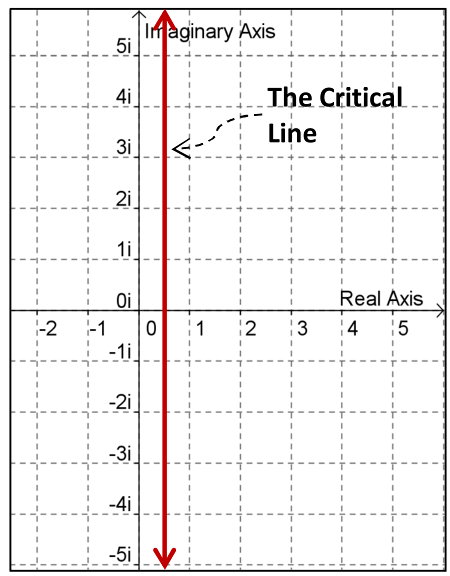
Through this process, it can be shown that $\zeta(-1) = -\frac{1}{12}$. Note that this conclusion is arrived upon using the sophisticated procedure of analytic continuation with functions of a complex argument and does not imply that the sum of all natural numbers is $-\frac{1}{12}$. The number to which the zeta function converges for a given value of *s* can be calculated easily using a computer. In fact, there also exists an explicit formula, in the form of an integral, for calculating this limit.



The Riemann Hypothesis

The famous Riemann hypothesis is concerned with the zeros of the Riemann zeta function. That is, it comes from the question, "For what values of *s* does $\zeta(s) = 0$?" The obvious solutions, known as trivial solutions, are $s = -2, -4, -6, \ldots$ (all even negative integers). The difficulty lies in finding other, nontrivial, zeros. It has been shown that the nontrivial zeros must lie in the strip between the vertical line through 0 and the vertical line through 1 (not inclusive). This region is often referred to as the critical strip, as shown in the diagram on the left. The Riemann hypothesis proposes that all nontrivial zeros lie on the vertical line with real part $\frac{1}{2}$. This line is commonly referred to as the critical line, as shown in the diagram below.





In general,

> **The Riemann hypothesis suggests that all nontrivial zeros of the Riemann zeta function have a real part of $\frac{1}{2}$.**

Billions of zeros that lie on the critical line have been found. No nontrivial zeros have yet been located in the critical strip that are not on the critical line and it has been proven that at least 40% of the zeros of the Riemann zeta function must have a real part of $\frac{1}{2}$ (Hawking, 2007). Although Riemann eventually abandoned his hypothesis, many other mathematicians have made efforts to prove it, including Hilbert, Hardy, Littlewood, Ramanujan, Siegel, Selberg, Erdős and Connes. Advances in computer technology have contributed greatly to the further investigation of the hypothesis, but no proof has been found to date.

The Riemann Hypothesis In Mathematics

Before discussing the application of the Riemann hypothesis to other disciplines, let's first touch upon its huge importance in the subject of mathematics. The significance of the hypothesis is most evident in the field of number theory, in which mathematicians study the properties and relationships of the natural numbers (1, 2, 3, 4, …). It is particularly relevant in the study of prime numbers. When looking at the occurrence of primes among the natural numbers, it is not at all obvious when the next prime number will occur. That is, the distribution of the prime numbers seems random. Riemann explained that the behaviour and location of the zeros of the zeta function has a direct relationship to the distribution of prime numbers. In fact, he developed a formula for determining the number of prime numbers between 1 and any other number. This formula, however, assumes that the Riemann hypothesis is true. Furthermore, mathematicians have since built many other propositions on the assumption that the Riemann hypothesis is true. It is for this reason that in his book *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics*, Marcus du Sautoy states, "The person who proves the Riemann hypothesis will have made it possible to fill in the missing gaps in thousands of theorems that rely on it being true." (du Sautoy, 2003) A proof of the Riemann hypothesis would undoubtedly allow mathematicians to make deep conclusions about prime numbers

To catch a glimpse of how the Riemann hypothesis and the Riemann zeta function are connected to the characteristics of prime numbers, we can once again turn to the work of Leonhard Euler. Euler made the connection between the infinite series $\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots$, which involves all of the positive integers, and the prime numbers by writing the series as the infinite products of the infinite geometric series for each prime number p, as follows:

$$\left\{1+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\right\}\times\left\{1+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right\}\times\left\{1+\frac{1}{5^1}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots\right\}\times\ldots\times\left\{1+\frac{1}{p^1}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots\right\}\times\ldots$$

That is, instead of writing the series as an infinite addition, it can be written as a multiplication. This property is based on the fact that that each integer can be factored into prime numbers in exactly one way. For example,

$$\frac{1}{60}=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{5}=\left(\frac{1}{2}\right)^2\times\frac{1}{3}\times\frac{1}{5}$$

Using the fact that $1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\ldots=\frac{1}{\left(1-p^{-s}\right)}$, the Riemann zeta function can be written as

$\zeta(s)=\prod_p\frac{1}{1-p^{-s}}$, where $\prod_p$ represents the product using all prime numbers *p*. Based on this representation, it can be shown that the zeta function converges only for complex numbers that have a real part greater than 1. Recall that Riemann extended this idea to all complex numbers except for *s* = 1.

In addition to unlocking the mystery of prime number distribution, a proof of the Riemann hypothesis would connect two completely different branches of mathematics; number theory, which deals with integers, and complex analysis, which deals with complex numbers. It is very surprising that the two fields are so closely connected.

THE RIEMANN HYPOTHESIS IN BUSINESS AND ENCRYPTION

Perhaps surprisingly, the Riemann hypothesis holds value in disciplines other than pure mathematics. To see how the hypothesis is significant in the business world, we simply need to consider the use of electronic transactions. Named after Ron Rives, Adi Shamir and Leonard Adleman, who first described the system in 1977, RSA is a method of data encryption used to secure electronic transactions. This system is based on the use of large prime numbers to protect credit card information and other electronic data such as that shared on the internet. Specifically, it makes use of the fact that it is very easy to build codes using large prime numbers, but very difficult to break them. The reason it is so difficult to break such codes results from the difficulty in factoring an extremely large number into its product of primes due to the seemingly random occurrence of prime numbers. Clearly, the more that is discovered about the nature of primes, the less secure the codes that make use of them will become. As a result, companies such as AT&T and Hewlett-Packard have dedicated significant amounts of money to research on prime numbers and the Riemann hypothesis (du Sautoy2003). If the Riemann hypothesis were proven, mathematicians could easily locate prime numbers with any number of digits, thereby facilitating the cracking of RSA codes and threatening the security of electronic transactions.

THE RIEMANN HYPOTHESIS IN QUANTUM PHYSICS

The Riemann hypothesis has also made an appearance in the field of quantum physics. In the 1920s, physicists noted that there is a relationship between the energy levels at which electrons vibrate in an atom and the mathematics describing the frequencies in the sound of a drum. Specifically, the forces that govern the vibrations of subatomic particles are analogous to the characteristics of a drum skin (tension, surrounding air pressure, etc.) that contribute to its sound when struck. As mathematics can be used to describe the waveforms that are created on the skin of a drum, so can mathematics account for the behaviour of the atom. To solve the equations involved in the physics of the atom, however, imaginary numbers must be used. It is these imaginary numbers that explain atomic phenomenon we cannot observe, such as an unobserved electron's ability to be in two places simultaneously or vibrating at several different frequencies. Marcus du Sautoy writes, "When we observe an event in the quantum world, it is as though we are seeing not the event itself in its natural domain, but a shadow of the event projected into our 'real' world of ordinary numbers. The act of observation causes the two-dimensional imaginary world to collapse into the one-dimensional line of ordinary numbers." (du Sautoy, 2003)

Physicists Werner Heisenberg and Max Born were crucial figures in the development of the quantum drum model of the atom, but it was the German mathematician David Hilbert who initially considered the possibility that the zeros of the Riemann zeta function were related to the mathematics of the vibrations in these drums. The idea was later revisited by American mathematician Hugh Montgomery, who, after a chance discussion with British physicist Freeman Dyson, realized that the characteristic frequencies that correspond closest with the zeros of the zeta function came from some of the most complicated atoms. Although simple atoms like hydrogen could be linked to a drum using fairly manageable equations, more complicated atoms required the use of statistics, which is how Eugene Wigner and Lev Landau approached the problem of describing atomic energy levels in the 1950s. They investigated whether or not the nucleus of a heavy atom, such as uranium, behaved like the majority of other quantum drums and found that indeed it did. Remarkably, the gaps between the zeros of the Riemann function and the gaps between energy levels showed statistical similarities. Although confirming these similarities was beyond the scope of Montgomery's computational power, Andrew Odlyzko later used a supercomputer to investigate the relationship. In 1989 he plotted the distances between zeros of the zeta function (up to 10^{20}) and compared it to the predictions made by Montgomery. The results were very similar, but as Odlyzko continued his investigation, he began to witness discrepancies. The University of Bristol's Sir Michael Berry took Odlyzko's work to the next level using quantum chaos theory.

Chaos refers to small changes of an input resulting in dramatic changes of the output. It also refers to systems for which almost all initial conditions will eventually result in a complete coverage of all possible outcomes. To visualize the concept of chaos, imagine the path of a billiard ball on a strangely shaped pool table. A small change in the initial striking of the ball will have a large impact on its path. Furthermore, if the surface of the table was frictionless and the ball was allowed to continue reflecting off the table's sides, the trace of its path would eventually cover the entire surface of the table. Quantum chaos is a branch of physics that considers the relationship between chaotic classical dynamic systems and quantum theory. The goal of the study of quantum chaos is to determine how quantum mechanics (the physical behaviour of matter and energy on atomic and subatomic levels) can describe classical mechanics (the motion of larger bodies based Newton's laws of motion). Berry compared the zeros of the Riemann zeta function to the energy levels found in chaotic quantum billiards. That is, using semiconductors, he analyzed the energy levels of a free electron tracing out a path on a tiny billiard table of predetermined shape and found that they matched the zeta function's zeros perfectly. When these electrons are confined to a rectangular shaped table, their paths are non-chaotic and the energy levels are randomly distributed, but when a stadium-shaped table is used and the paths are chaotic, the energy levels form a much more uniform pattern and no two energy levels are close to each other. The strong connections between the Riemann hypothesis and energy levels is intriguing and hints that perhaps a proof will eventually come from the discipline of physics.

Conclusion

The Riemann hypothesis has demanded the attention of mathematicians since its public introduction in 1859. From its intimate relationship with prime numbers to its connections in science and technology, the importance of the hypothesis cannot be overstated. As more mathematics is developed on the assumption of its truth, and as it continues to intertwine itself with other disciplines, the desire to turn the Riemann hypothesis into the Riemann theorem will undoubtedly grow.

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