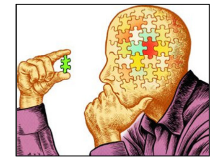
ANTIDIFFERENTIATION BY SUBSTITUTION

INDEFINITE INTEGRALS

Recall that the family of antiderivatives for a function $f(x)$ is $F(x) + C$, where $F'(x) = f(x)$.

Notice that we have not yet considered a convenient notation for working with antiderivatives.

Since the Fundamental Theorem of Calculus gives us a powerful link between antiderivatives and integration, we use the following name and notation:



DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

CAUTION

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function (or family of functions).

Example (*complete on a separate page*)

Evaluate each of the following.

a) $\int (x^3 + 6x + 1)dx$ b) $\int x^{-\frac{3}{4}}dx$ c) $\int (x^2 - \sin x)dx$

d) $\int (1-t)(2+t^2)dt$ e) $\int \cos 4u\, du$

Properties of Indefinite Integrals

$$\int k f(x)dx = k\int f(x)\,dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1 \qquad \int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$$

Trigonometric Formulas

$$\int \cos u\,du = \sin u + C \qquad \int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C \qquad \int \csc u \cot u\,du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

Example (*complete on a separate page*)

Use differentiation to verify each of the following antiderivative formulas.

a) $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$

b) $\int \ln u\,du = u\ln u - u + C$

Question for Discussion

Why do we use the absolute value here?

A Note on Leibniz Notation

We know that the differential "dx" in the definite integral $\int_a^b f(x)\,dx$ is related to the Riemann sum, but is it necessary in the indefinite integral $\int f(x)\,dx$?

Are $\int f(u)\,du$ and $\int f(u)\,dx$ the same thing?

The following exploration will help realize the importance of the differential.

EXPLORATION **Are $\int f(u)\,du$ and $\int f(u)\,dx$ the Same Thing?**

Let $u = x^2$ and let $f(u) = u^3$.

1. Find $\int f(u)\,du$ as a function of u.
2. Use your answer to question 1 to write $\int f(u)\,du$ as a function of x.
3. Show that $f(u) = x^6$ and find $\int f(u)\,dx$ as a function of x.
4. Are the answers to questions 2 and 3 the same?

Example (*complete on a separate page*)

Let $f(x) = x^3 + 1$ and let $u = x^2$. Find each of the following antiderivatives in terms of x.

a) $\int f(x)\,dx$ b) $\int f(u)\,du$ c) $\int f(u)\,dx$

SUBSTITUTION IN INDEFINITE INTEGRALS

In order to evaluate an integral, we occasionally make a variable substitution (often called a "*u*-substitution").

This substitution can turn an unfamiliar integral into a familiar one that we can easily evaluate.

An Example

Evaluate $\int \sin x\, e^{\cos x}\,dx$.

Solution

$$\int \sin x\, e^{\cos x}\,dx$$
$$= \int e^{\cos x} \sin x\,dx$$
$$= \int e^{\cos x}[-(-\sin x)]\,dx$$
$$= -\int e^{\cos x}(-\sin x)\,dx$$
$$= -\int e^{u}\,du$$
$$= -e^{u} + C$$
$$= -e^{\cos x} + C$$

Let $u = \cos x$

Then $\frac{du}{dx} = -\sin x$

and thus $du = -\sin x\,dx$

Questions for Discussion

How do we know when a substitution is required?

How do we decide upon the substitution to be used?

Example (*complete on a separate page*)

Evaluate the following indefinite integrals.

a) $\int x^3 \cos(x^4 + 2)\,dx$ b) $\int \sqrt{2x+1}\,dx$

c) $\int x^2\sqrt{5 + 2x^3}\,dx$ d) $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Sometimes a trigonometric identity is helpful...

e) $\int \frac{dx}{\cos^2 2x}$ f) $\int \cot^2 3x\,dx$ g) $\int \cos^3 x\,dx$

The method of substitution can also be useful when evaluating definite integrals.

Example (*complete on a separate page*)

Evaluate the following definite integrals.

a) $\int_0^{\frac{\pi}{3}} \tan x \sec^2 x\,dx$ b) $\int_0^1 \frac{x}{x^2 - 4}\,dx$