The Constant Rule

Derivative of a Constant Function:
The Constant Rule

If $f(x)$ is a constant function, that is, $f(x) = c$, then $f'(x) = 0$.

In Leibniz notation, $\frac{d}{dx}(c) = 0$.

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} 0$$

$$= 0 \quad \blacksquare$$

The Power Rule

Derivative of a Power Function:
The Power Rule

If n is a positive integer and $f(x) = x^n$, then $f'(x) = nx^{n-1}$. In Leibniz notation, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof:

Before we prove the power rule, we will establish the identity $x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1})$, which will be used in the proof.

The sequence $x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1}$ is a geometric sequence of n terms, where the first term is x^{n-1} and the common ratio is $\frac{a}{x}$.

Recall that the sum of a geometric series is

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{x^{n-1}\left[1 - \left(\frac{a}{x}\right)^n\right]}{1 - \frac{a}{x}} = \frac{x^{n-1}\left[\frac{x^n - a^n}{x^n}\right]}{\frac{x - a}{x}} = \frac{x^n - a^n}{x - a}$$

Therefore,

$$(x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1}) = (x - a)\frac{x^n - a^n}{x - a}$$

$$= x^n - a^n$$

Use this result for $f(x) = x^n$:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{[(x + h) - x][(x + h)^{n-1} + (x + h)^{n-2}x + \ldots + x^{n-1}]}{h}$$ Apply the identity proven above.

$$= \lim_{h \to 0} \frac{\not{h}^1[(x + h)^{n-1} + (x + h)^{n-2}x + \ldots + x^{n-1}]}{\not{h}^1}$$

$$= x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \ldots + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + x^{n-1} + \ldots + x^{n-1}$$ (n terms)

$$= nx^{n-1}$$

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The Constant Multiple Rule

The Constant Multiple Rule

If $g(x) = cf(x)$ and $f(x)$ is differentiable, then $g'(x) = cf'(x)$.
Equivalently, $\frac{d}{dx}[cf(x)] = c\left[\frac{d}{dx}f(x)\right]$.

Proof:

$$
\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h \to 0} c\left[\frac{f(x+h) - f(x)}{h}\right] \\
&= c \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right] \\
&= cf'(x) \quad \blacksquare
\end{aligned}
$$

Intuitively, if g is c times as big as f, then g increases or decreases c times as fast as f.

The Sum and Difference Rules

The Sum Rule

If $h(x) = f(x) + g(x)$ and f and g are both differentiable, then $h'(x) = f'(x) + g'(x)$. In Leibniz notation,
$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.

Proof:

Let $F(x) = f(x) + g(x)$.

$$h'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x) \quad \blacksquare$$

Note: The proof for the difference rule is similar to the proof for the sum rule.