SLOPE FIELDS AND EULER'S METHOD

DIFFERENTIAL EQUATIONS

Definition

An equation involving a derivative is called a **differential equation**.

Definition

The **order of a differential equation** is the order of the highest derivative involved in the equation.

We can often use our knowledge of antiderivatives to solve differential equations.

Example (*complete on a separate page*)

Find all functions y that satisfy $\frac{dy}{dx} = \sec^2 x + 2x + 5$.

INITIAL VALUE PROBLEMS

In the previous example, notice how the solution is an entire *family* of functions.

In order to find a unique solution to a differential equation, we must be given more information.

- As long as the solution to a first order differential equation is continuous, all we need is a single point on its graph.
- This point is called an **initial condition**.
- A differential equation with an initial condition is called an **initial value problem**.
- The unique solution to an initial value problem is called the **particular solution** to the differential equation.

Example (*complete on a separate page*)

Find the particular solution to the equation $\frac{dy}{dx} = e^x - 6x^2$ whose graph passes through the point $(1, 0)$.

Notice how the initial condition "pins down" the solution curve.

As long as the solution curve is continuous, the initial value will pin it down over its entire domain.

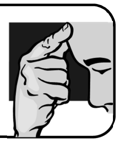
What happens if the solution curve has a discontinuity?

If the solution curve is discontinuous, the initial condition only pins down the piece of the curve that passes through the given point.

- In this case, we must specify the domain of the solution.

Question for Discussion

How is it possible for the initial condition to pin down only a piece of the solution curve? Can the same result occur when the solution curve is continuous?



Example (*complete on a separate page*)

Find the particular solution to the equation $\frac{dy}{dx} = 2x - \sec^2 x$ whose graph passes through the point $(0, 3)$.

Example

Use a calculator to graph seven of the functions that solve the differential equation $\frac{dy}{dx} = \cos x$.
How are these graphs related?

Sometimes we are unable to find an antiderivative to solve an initial value problem, but we can still find a solution using the Fundamental Theorem of Calculus.

Recall:

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Example *(complete on a separate page)*
Find the solution to the differential equation $f'(x) = e^{-x^2}$ for which $f(7) = 3$.

SLOPE FIELDS

By thinking about slopes, we can produce the family of solution curves for a differential equation without actually solving the differential equation.

Consider the differential equation $\frac{dy}{dx} = \cos x$.

We know that $\frac{dy}{dx}$ gives slope.

When x=0, we get $\frac{dy}{dx} = \cos(0) = 1$.

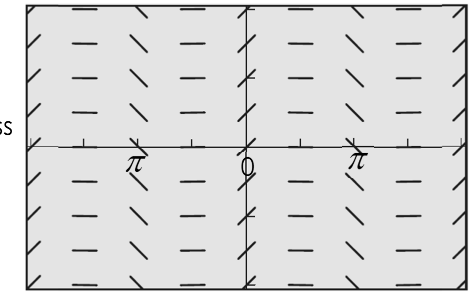
So, the slope at x=0 is 1, regardless of the *y*-value.

We can illustrate this slope using small line segments.

Similarly, at π and $-\pi$, $\frac{dy}{dx} = -1$.

At all odd multiples of $\pi/2$, $\frac{dy}{dx} = 0$.

At 2π and -2π, $\frac{dy}{dx} = 1$.

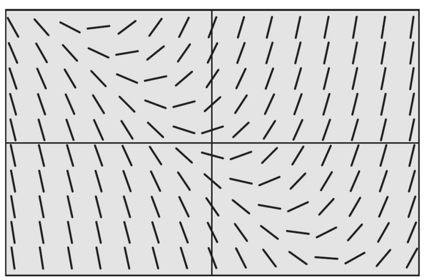


The above graph, called a **slope field**, gives an approximation of the differential equation's family of solution curves.

Another Example

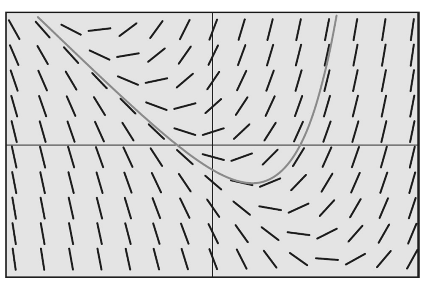
Consider the differential equation $\frac{dy}{dx} = x + y$.

Use your calculator to construct the slope field for the differential equation.



Notice the following properties:

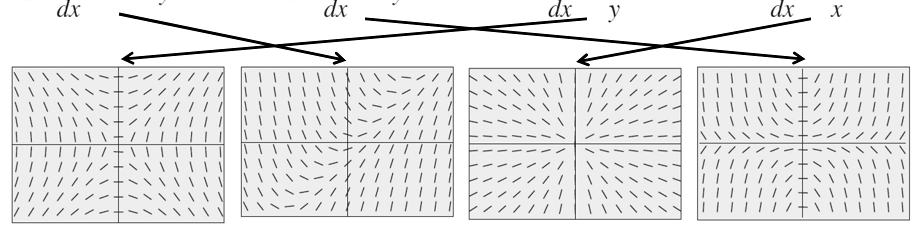
- The slopes are zero along the line $x + y = 0$.
- The slopes are -1 along the line $x + y = -1$.
- The slopes get steeper as x increases.
- The slopes get steeper as y increases.
- The lower graph shows the particular solution that passes through the point (2, 0).



Example

Use slope analysis to match the following differential equations to the given slope fields. Do not use your calculator.

1. $\frac{dy}{dx} = x - y$ **2.** $\frac{dy}{dx} = xy$ **3.** $\frac{dy}{dx} = \frac{x}{y}$ **4.** $\frac{dy}{dx} = \frac{y}{x}$

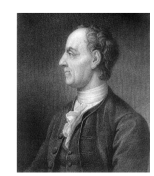


(a) (b) (c) (d)

EULER'S METHOD

Earlier, we graphed a particular solution to a differential equation by first producing a slope field.

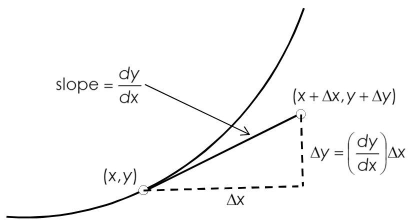
We can actually graph the particular solution directly by starting at the given point and piecing together little line segments to build a continuous approximation of the curve.

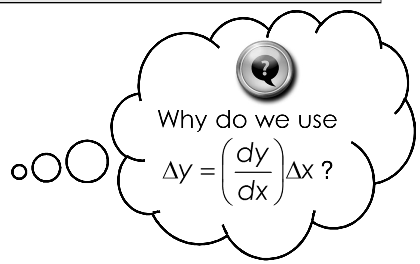


Leonhard Euler
1707 - 1783

Euler's Method For Graphing a Solution to an Initial Value Problem

1. Begin at the point (x, y) specified by the initial condition. This point will be on the graph, as required.
2. Use the differential equation to find the slope dy/dx at the point.
3. Increase x by a small amount Δx. Increase y by a small amount Δy, where $\Delta y = (dy/dx)\Delta x$. This defines a new point $(x + \Delta x, y + \Delta y)$ that lies along the linearization.
4. Using this new point, return to step 2. Repeating the process constructs the graph to the right of the initial point.
5. To construct the graph moving to the left from the initial point, repeat the process using negative values for Δx.





Examples (*complete on a separate page*)

1) Let f be the function that satisfies the initial value problem $dy/dx = x + y$ with $f(2)=0$. Use Euler's Method and increments of $\Delta x = 0.2$ to approximate $f(3)$. Use a table with the headings shown below to record your calculations.

(x, y)	$dy/dx = x + y$	Δx	$\Delta y = (dy/dx)\Delta x$	$(x + \Delta x, y + \Delta y)$

2) If $dy/dx = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's Method with five equal steps to approximate y when $x = 1.5$. Use a table with the headings shown above to record your calculations.

3) When, in general, does Euler's Method give an underestimate and when does it give an overestimate?