Function Study Notes ! !

Unit 1 - Functions

Function - a rule that states for every output there is one input
x can only have one y value

Vertical Line Test - if any vertical line passes through more than one point on the graph then it is NOT a function

Function Notation:

$$\underbrace{f(x)} = \pm a(\pm b(x-h)) \pm k$$

"f at x" f is a function of x

Does not mean f times x!

Domain - the set of INPUTS in a relation (all the x values)

Range - the set of OUTPUTS in a relation (all the y values)

Absolute Value - the DISTANCE that a number is from zero
the absolute value is always POSITIVE

Absolute Value Notation:

$$|x|$$

Writing Absolute Value As A Piecewise Function:

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Base Function - refers to the original function before any transformations have been applied to it

Intervals of Increasing - when the graph rises from left to right
$f(x_1) < f(x_2)$ whenever $x_1 < x_2$

Intervals of Decreasing - when the graph falls from left to right
$f(x_1) > f(x_2)$ whenever $x_1 < x_2$

Absolute Maximum - highest point on the graph
$f(a) \geq f(x)$ for all of x in the domain of f

Absolute Minimum - lowest point on the graph
$f(b) \leq f(x)$ for all of x in the domain of f

Local Maximum - when a function changes from increasing to decreasing

Local Minimum - when a function changes from decreasing to increasing

Note: Maximum or minimum values that occur at endpoints are NOT considered to be local maximums or local minimums

Continuity - a continuous function does not contain any holes or breaks over its entire domain

a discontinuous function does contain wholes and breaks which can occur anywhere throught domain

Symmetry - an even function symmetric about the y-axis

an odd function has rotational symmetry about the origin

Algebraically Solving Symmetry:

$$f(-x) = f(x)$$

replace all "x" with "-x" and still result in the original $f(x)$ function

Even Symmetry

$$f(-x) = -f(x)$$

replace all "x" with "-x" and result in opposite signs of the original $f(x)$ function

Odd Symmetry

End Behaviour - refers to what happends as the function x-values become very large positively or very large negatively

otherwise known as the function x-values approach infinity + or as the function x-valuse approach infinity -

End Behaviour Notation:

As $x \to \infty$, $f(x) \to$

As $x \to -\infty$, $f(x) \to$

Transformation Equation:

$$f(x) = \pm a(\pm b(x \pm h)) \pm k$$

a: vertical stretch or compression

b: horizontal stretch or compression

h: horizontal shift right or left

k: vertical shift up or down

$\pm$: infront of a or b is reflection

$\pm$: infront of h is left or right

$\pm$: infront of k is up or down

Inverse Relations - the inverse of a function f is the function that "UNDOES" f

Inverse Of A Function Notation:

$$f^{-1}$$

Piecewise Functions - a function defined by using different rules on different intervals

consists of several pieces all of which are functions

Piecewise Notation:

$$f(x) = \{$$

Exploring Operations With Functions -

when adding two functions we add the two outputs values for every valid input

We can only add where their domains overlap

The same ideas apply to subtraction and multiplication

Unit 2 - Rates Of Change

Distance - the total distance travelled by an object
NEVER a negative value

Displacement - an objects position relative to a fixed point
can be a negative value

Speed - how fast an object is travelling without regard to direction
NEVER a negative value

Velocity - how fast an object is traveling and the direction of its motion
can be a negative value

Displacement - Time Graph - graphically displays and describes an objects position relative to a fixed point

Velocity-Time Graph - graphically displays and describes how fast an object is travelling and its direction
found by finding the slopes of a displacement-time graph

Note: velocity equals zero on the max and min on a displacement - time graph
when the displacement-time graph is a straight line the velocity-time graph will be a horizontal line to represent a constant speed
when the displacement - time graph is a horizontal line the velocity-time graph will show a velocity of zero
positive slopes on the displacement-time graph means positive values on velocity-time graphs and same thing with negative slopes

Average Rate of Change - calculates the amount of change in one item divided by the corresponding amount of change in another

i.e change in distance over change in time

otherwise described as slope

Average Rate of Change:

$$A.R.O.C = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{slope}$$

Instantaneous Rate of Change - the rate of change at a particular moment

I.R.O.C of a function at a given point is the slope of the tangent line at that point

The Tangent Line - touches exactly one point of the graph

Slope of A Tangent - the slope of tangent at P is the limiting value of the slopes of the secants PQ as Q approaches point P

Note: if we want the rate of change of a function at a particular point we are really after the slope of the tangent line at that point

cannot find the slope of the tangent must use two methods to find it, which gives us the I.R.O.C

Method 1 To Find I.R.O.C:

We investigate the slopes of the secant (A.R.O.C) through the point of interest and a set of points closely approaching the point of interest from both the left and the right

i.e $f(x) = 3x^2 - 2x + 5$ where $x = 4$

From Left

interval	Δx	$\Delta f(x)$	$M_{secant}\left(\frac{\Delta f(x)}{\Delta x}\right)$
$3.9 \le x \le 4$	$4 - 3.9 = 0.1$	$f(4) - f(3.9) = 2.17$	$\frac{2.17}{0.1} = 21.7$
$3.99 \le x \le 4$	0.01	0.2197	21.97
$3.999 \le x \le 4$	0.001	0.021997	21.997
$3.9999 \le x \le 4$	0.0001	0.00219997	21.9997

From Right

interval	Δx	$\Delta f(x)$	$M_{secant}\left(\frac{f(x)}{x}\right)$
$4 \le x \le 4.1$	0.1	2.23	22.3
$4 \le x \le 4.01$	0.01	0.2203	22.03
$4 \le x \le 4.001$	0.001	0.022003	22.003
$4 \le x \le 4.0001$	0.0001	0.0022003	22.0003

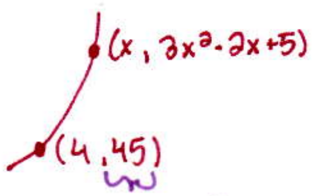
$\therefore$ we see secant slopes approach the value of 22

$\therefore$ tangent line @ $x = 4$ is approx 22

Method 2 To Find I.R.O.C :

We create a formula for the slope of the secant line through the point of interest and a general point on the curve

i.e $f(x) = 3x^2 - 2x + 5$ where $x = 4$



found by subbing in 4 into $f(x)$

$$m_{secant} = \frac{\Delta f(x)}{\Delta x}$$

$$= \frac{3x^2 - 2x + 5 - 45}{x - 4}$$

$$= \frac{3x^2 - 2x - 40}{x - 4}$$

$$= \frac{(3x+10)\cancel{(x-4)}}{\cancel{x-4}}$$

$$= 3x + 10, \ x \neq 4$$

Will always factor and cancel. You do not need to do that though!

From Left

x	Msecant
3.9	21.7
3.99	21.97
3.999	21.997
3.9999	21.9997

From Right

x	Msecant
4.1	22.3
4.01	22.03
4.001	22.003
4.0001	22.0003

∴ we see the secant slopes approach a value of 22

∴ tangent slopes @ x=4 is approx 22

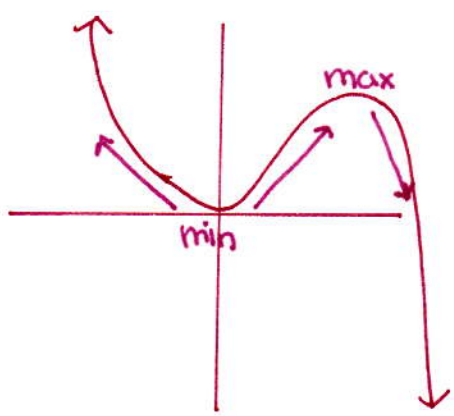
Maximum and Minimum Values -

the slope of the tangent where a local max and min occurs is zero

Determing Maximums and Minimums :

We can determine minimums when the slope of the secant goes from a negative value, from the left side of a specific point, to a positive secant slope, from the right side. These values are also small numbers

We can determine maximums when the slope of the secant goes from a positive value to a negative secant value. These values are also large numbers



Unit 3- Polynomial Functions

Polynomial Functions - an expression that has only powers in it
it must be whole numbers
each of these powers will have a coefficient that can be any number you can think of

i.e - $5x^7 - 3x^2 + 7$ $\quad 3x^2 + 7y^2 + 1$

The Degree - the value of the highest exponent of the variable

Turning Point - a point on a curve that is higher or lower than all nearby points

Zeros of Polynomial Functions in Factored Form:

The zero of a linear function f(x) is s in $f(x) = k(x-s)$
The zeros of a quadratic function f(x) are s and t in $f(x) = k(x-s)(x-t)$
The zeros of a cubic function f(x) are s, t and u in $f(x) = k(x-s)(x-t)(x-u)$

Note: if you want the zeros factor the polynomial and set each factor to zero

Characteristics of Polynomial Functions:

A polynomial of degree n has n-1 possible turning points

i.e - 55 factors = 54 possible turning points

Order - the exponent to which a factor in an algebraic expression is raised

i.e - $f(x) = (x-3)^2(x-1)$ order of $(x-3)$ is 2, the order of $(x-1)$ is 1

Family - a set of polynomial functions whose equations have the same degree and whose graphs have common characteristics

Diving Polynomials:

We can divide polynomials using two different methods.

Method 1: Long Division

Find $(x^2+7x-3) \div (x+1)$

$$\begin{array}{r@{\,}l}
 & \phantom{x^2+{}}x+6 \\
x+1 & \overline{)\,x^2+7x-3} \\
 & \underline{x^2+x} \downarrow \\
 & \phantom{x^2+{}}6x-3 \\
 & \phantom{x^2+{}}\underline{6x+6} \\
 & -9
\end{array}$$

$x^2+7x-3 = (x+1)(x+6)-9$

dividend = divisor • quotient + remainder

Note: the remainder must be a smaller degree than the quotient
degree zero is lower than degree one

Find $(x^3-5x+9) \div (x-2)$

$$\begin{array}{r@{\,}l}
 & \phantom{x^3+{}}x^2+2x-1 \\
x-2 & \overline{)\,x^3+0x^2-5x+9} \\
 & \underline{x^3-2x^2} \downarrow \\
 & \phantom{x^3+{}}-2x^2-5x \\
 & \phantom{x^3+{}}\underline{2x^2-4x} \downarrow \\
 & -x+9 \\
 & \underline{-x+2} \\
 & \phantom{x^3+0x^2-5x+{}}7
\end{array}$$

$x^3-5x+9 = (x-2)(x^2+2x-1)+7$

Note: the dividend did not have an x^2 value. To solve the problem you add zero x^2
You are adding nothing therefore not changing the question

Method 2: Synthetic Division

A simple version to divide by dealing with just the coefficents

Steps:

1. Decide on the k value
2. List the coefficients in the dividend
3. Bring down the first coefficient
4. Multiply and add to get the rest of the coefficients
5. Write out the results using the variable powers

Find $(4x^3-5x^2+3x-7) \div (x-\underline{2})$

k value = $\underline{2}$

$$\begin{array}{c|cccc}
2 & 4 & -5 & 3 & -7 \\
 & \downarrow & 8 & 6 & 18 \\
\hline
 & 4 & 3 & 9 & 11 \\
 & x^2 & x & \text{constant} & \text{remainder}
\end{array}$$

$4x^2+3x+9$ R 11

$4x^3-5x^2+3x-7 = (x-2)(4x^2+3x+9)+11$

Method 2 Continued:

Find $(12x^3+2x^2+11x+16) \div (3x+2)$

K value $= -2/3$

$12x^3+2x^2+11x+16 = (3x+2)(4x^2-2x+5)+6$

$-\frac{2}{3}$	12	2	11	16
		-8	4	-10
	12	-6	15	6
$\div 3$		$\div 3$	$\div 3$	remainder
	4	-2	5	6
	x^2	x	constant	remainder

Note: method 2 only works on degree one polynomials (when the divisor is degree one)
the K value is the opposite sign of the divisor
when you have a fraction as your K value you must divide a second time by the denominator
you do not divide the remainder a second time b/c the remainder has not fully been divide the first time

Factoring Polynomials:

We know for a polynomial $P(x)$: if we can find a number K such that $P(K)=0$ then we will know that the remainder, when we divide $P(x)$ by $x-K$, will be zero
$\therefore$ $x-K$ is a factor of $P(x)$

The Remainder Theorem - when a polynomial $P(x)$ is divided by $x-K$ the remainder is equal to $P(K)$
when a polynomial $P(x)$ is divided by $jx-K$ the remainder is equal to $P(\frac{k}{j})$

The Factor Theorem - $x-K$ is a factor of $P(x)$ if and only if $P(K)=0$
$jx-k$ is a factor of $P(x)$ if and only if $P(\frac{k}{j})=0$

Steps:

1. Find a K value such that $P(K)=0$
2. Now you know that $(x-K)$ is a factor of $P(x)$ so divide $P(x)$ by $(x-K)$ to find the other
3. factor
4. Repeat the process until you get all the factors down to degree 2
5. Factor any degree 2 factors

Factor $2x^4 - 9x^3 - x^2 + 18x + 8$

Let $P(x) = 2x^4 - 9x^3 - x^2 + 18x + 8$

$P(-1) = 0$

$\therefore$ $x + 1$ is a factor

k value $= -1$

$\times -1$	2	-9	-1	18	8
	↓	-2	11	-10	-8
	2	-11	10	8	☺

$\therefore P(x) = (x+1)(2x^3 - 11x^2 + 10x + 8)$

Let $P(x) = 2x^3 - 11x^2 + 10x + 8$

$P(2) = 0$

$\therefore (x-2)$ is a factor

k value $= 2$

$\times 2$	2	-11	10	8
	↓	4	-14	-8
	2	-7	-4	☺

$\therefore P(x) = (x+1)(x-2)(2x^2 - 7x - 4)$

$P(x) = (x+1)(x-2)(2x+1)(x-4)$

Note: If the equation you need to factor does not consists of all ordered degrees i.e $7x^4 + 2x^2 + x - 2$, when you divide you must add zeros of that degree x.

Factoring Polynomials with Extra Variables:

Find k such that x - 3 is a factor of $f(x) = 2x^3 - x^2 + kx + 36$

if $x - 3$ is a factor then $f(3) = 0$

$0 = 2(3)^3 - (3)^2 + k(3) + 36$

$= 54 - 9 + 3k + 36$

$3k = -81$

$k = -27$

$k = 3$

$\times 3$	2	-1	k	36
	↓	6	15	-36
	2	5	-12	☺

$k + 15 = -12$

$k = -27$

Graphing Polynomials:

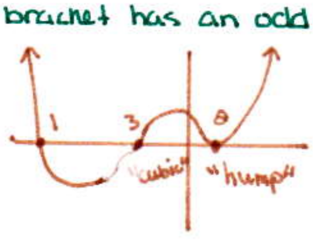
Steps:

Factor the polynomials if they are not already factored

Each factor is where the graph is zero so plot the points, the zeros

If the factor bracket has an even number on it, the graph makes a "hump" at that zero

If the factor bracket has an odd number on it, the graph looks cubic at that zero



$y = (x + \_)^3 (x - \_)^2 (x + \_)^1$

Factoring a Sum or Difference of Cubes:

$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$

$$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$$

Leading Coefficient - the term of highest power of x

Unit 4 - Polynomial Equations and Inequalities

Steps to Solving Polynomial Functions:

- Substitute in the value required for the function
- Move all terms to one side so that one side equals zero
- Factor and solve for the values in the requested domain

Solve for positive x values that make $P(x) = 10$ in the polynomial $P(x) = x^4 + x^3 - 16x^2 + 26x - 2$

Set $P(x) = 10$

$10 = x^4 + x^3 - 16x^2 + 26x - 2$

$0 = x^4 + x^2 - 16x^2 + 26x - 2 - 10$

$0 = x^4 + x^2 - 16x^2 + 26x - 12$

$P(x) = x^4 + x^3 - 16x^2 + 26x - 12$

$P(1) = 0$

$\therefore (x-1)$ is a factor

$h = 1$

× 1	1	1	-16	26	-12
	↓	1	2	-14	12
	1	2	-14	12	☺

$\therefore x^4 + x^3 - 16x^2 + 26x - 2 = (x-1)(x^3 + 2x^2 - 14x + 12)$

Let $P(x) = x^3 + 2x^2 - 14x + 12$

$P(2) = 0$

$\therefore (x-2)$ is a factor

$h = 2$

× 2	1	2	-14	12
	↓	2	8	-12
	1	4	-6	☺

$\therefore x^4 + x^3 - 16x^2 + 26x - 12 = (x-1)(x-2)(x^2 + 4x - 6)$

$\therefore 0 = (x-1)(x-2)(x^2 + 4x - 6)$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 + 24}}{2}$$

$$= \frac{-4 \pm \sqrt{40}}{2}$$

$x = 1.15$ or $x = \cancel{-5.15}$

not positive

$\therefore$ the positive solutions are $x = 1$ $x = 2$ $x = 1.15$

Solving Linear Inequalities :

A linear inequality is an inequality that contains an algebraic expression of degree 1

$<$ less than $\qquad$ $>$ greater than

We solve linear inequalities similar to the way we solve linear equations

Equation	Inequality
$3x-1=8$	$3x-1<8$
$3x=8+1$	$3x<8+1$
$3x=9$	$3x<9$
$x=\frac{9}{3}$	$x<\frac{9}{3}$
$=3$	$x<3$

Set Notation: $\{x\in\mathbb{R} \mid x<3\}$

Interval Notation: $x\in(-\infty,3)$

$$x-29<4x-5<x+12$$

$x-29<4x-5$
$-24<3x$
$-8<x$

$4x-5<x+12$
$3x<17$
$x<\frac{17}{3}$

$$-8<x<\frac{17}{3}$$

Solve $x^4-8x<0$

$x(x^3-8)<0$ $\qquad$ using the difference of cubes

$x(x-2)(x^2+2x+4)<0$

We want x values where

$x(x-2)(x^2+2x+4)=0$

$\therefore x=0 \quad x=2 \quad$ or $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-2\pm\sqrt{-12}}{2}$

no solutions

$\therefore$ no zeros

We want to know when $x<0$

Interval	$x<0$	$0<x<2$	$x>2$
$x(x-2)(x^2+2x+4)$	+	−	+

use the zeros to fill in the positive/negative chart

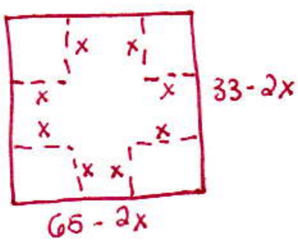
$\therefore x^4-8x<0$ when $0<x<2$

Rates of Change in Polynomial Functions :

The process of average rate of change and instantaneous rate of change is the same when dealing with inequalities

Word Problems :

A piece of metal 65cm by 33cm has congruent squares notched out of each corner to make it so the metal can be folded up into an open top box. If the box is to have a volume of 4779 cm^3, find the length, width and height of all possible boxes.



$$4779 = x(33-2x)(65-2x)$$
$$4779 = x(2145-66x-130x+4x^2)$$
$$4779 = x(4x^2-196x+2145)$$
$$0 = 4x^3-196x^2+2145x-4779$$

Let $f(x) = 4x^3-196x^2+2145x-4779$

$f(3) = 0$

×3	4	-196	2145	-4779
	↓	12	-552	4779
	4	-184	1593	☺

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{184 \pm \sqrt{184^2-4(4)(1593)}}{2(4)}$$
$$= \frac{184 \pm \sqrt{8368}}{8}$$

$x = 34.43459663$

or

$x = 11.56540337$

To find which x-values work

$65 - 2(3) = 59$ ✓
$33 - 2(3) = 27$

$65 - 2(34.4) = -3.8$ X
$33 - 2(34.4) = -35.8$

$65 - 2(11.6) = 41.8$ ✓
$33 - 2(11.6) = 9.8$

∴ possibilities of the dimensions are 59cm by 27cm and 9.8cm by 11.6cm.

Unit 5 - Rational Functions, Equations and Inequalities

Reciprocal Functions:

$$y = \frac{1}{dog}$$ dog = function

Steps to graphing reciprocal functions:

- State $\boxed{d}$omain and its restrictions — **D**on't
- state $\boxed{i}$ntercepts — **I**nvite
- state $\boxed{a}$symptotes (horizontal, vertical, oblique) — **A**my
- make a positive/negative $\boxed{i}$nterval chart (includes zeros and vertical asymptotes) — **I**nto

Rational Function - in the form of $h(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials

Note: $g(x) \neq 0$

Vertical Asymptote - the same as domain restrictions
where the graph does not exist

Horizontal Asymptote - occurs in three different scenarios;

$\frac{\text{lower degree}}{\text{higher degree}}$ = horizontal asymptote of $y=0$

$\frac{\text{higher degree}}{\text{lower degree}}$: no horizontal asymptote

$\frac{\text{same degree}}{\text{same degree}}$: divide coefficients to get horizontal asymptote

Oblique Asymptote - occurs when the polynomial on the numerator is one degree higher than the polynomial on the denominator

i.e. - $y = \frac{2x^{\boxed{3}} - x^2 + 3}{x^{\boxed{2}}}$ $\therefore$ oblique asymptote is $y = 2x-1$

$y = 2x - 1 + \frac{3}{x^2}$

Note: an oblique and horizontal asymptote cannot occur at the same time

Solving Rational Equations:

when solving rational equations with multiple fractions either clear the fraction or get a common denominator

Solve $\frac{x+3}{x-4} = \frac{x-1}{x+2}$, $x \neq 4$ $x \neq -2$

$$\frac{(x-\cancel{4})(x+2)}{1} \cdot \frac{x+3}{x-\cancel{4}} = \frac{(x-4)(x+\cancel{2})}{1} \cdot \frac{x-1}{\cancel{x+2}}$$

$$(x+2)(x+3) = (x-4)(x-1)$$
$$x^2+5x+6 = x^2-5x+4$$
$$10x+2 = 0$$
$$10x = -2$$
$$x = -\frac{1}{5}$$

Check

L.S	R.S
$\frac{-\frac{1}{5}+3}{-\frac{1}{5}-4} \times \frac{5}{5}$	$\frac{-\frac{1}{5}-1}{-\frac{1}{5}+2} \times \frac{5}{5}$
$= \frac{-1+15}{-1-20}$	$= \frac{-1-5}{-1+10}$
$= -\frac{14}{20}$	$= -\frac{6}{9}$
$= -\frac{2}{3}$	$= -\frac{2}{3}$

LS = RS

Solving Rational Inequalities:

We use the same strategy from before; Don't Invite Amy Into to solve the rational inequalities

Solve $\frac{7}{x-3} \geq \frac{2}{x+4}$

$$\frac{7}{x-3} - \frac{2}{x+4} \geq 0$$
$$\frac{7(x+4)-2(x-3)}{(x-3)(x+4)} \geq 0$$
$$\frac{5x+34}{(x-3)(x+4)} \geq 0$$

Domain: $x \neq 3$ $x \neq -4$

Intercepts: x-int

$$0 = \frac{5x+34}{(x-3)(x+4)}$$
$$0 = 5x+34$$
$$\frac{-34}{5} = x$$
$$x = -6.8$$

y-int

$$y = \frac{5(0)+34}{(0-3)(0+4)}$$
$$= \frac{34}{12}$$
$$= 2.8$$

Asymptotes: $x=3$ $x=-4$

$y=0$

$\therefore \frac{7}{x-3} - \frac{2}{x+4} \geq 0$ when

$-6.8 \leq x < -4$ and $x > 3$

Positive / Negative Intervals:

	$x < -6.8$	$-6.8 < x < -4$	$-4 < x < 3$	$x > 3$
$\frac{5(x+6.8)}{(x-3)(x+4)}$	−	+	−	+

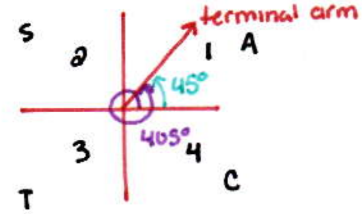
Rates of Change of Rational Functions:

The same methods of I.R.O.C is used when dealing with rational functions.

Unit 6- Trigonometric Functions

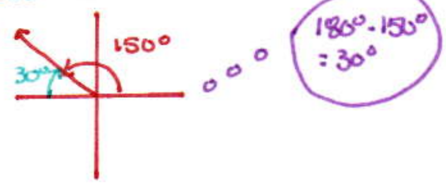
Coterminal Angles- when two angles in standard position have the same terminal arm

i.e



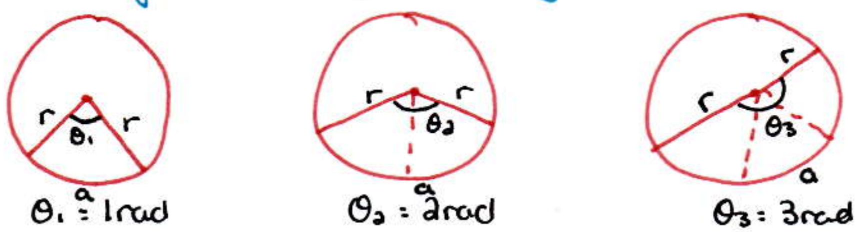
Related Acute Angle- an angle in standard position that is between the terminal arm an the x-axis
between 0° and 90°

i.e



R.A.A = 30°

Radian- the measure of an angle which is subtended at the centre of a circle by an arc equal in length to the radius of the circle



Radian Formula:

$$\text{number of radians} = \frac{\text{arc length}}{\text{radius}}$$

or

$$\theta = \frac{a}{r}$$

Converting Between Degrees and Radians:

To convert between degrees and radians you must know that 1 radian = 180° otherwise written as π rad = 180°

$$\pi \text{ rad} = \frac{180°}{\pi} \quad \text{and} \quad 1° = \frac{\pi}{180°}\text{rad}$$

i.e - $\frac{\pi}{3}\text{rad} \times \frac{180°}{\cancel{\pi \text{rad}}} = \frac{180°}{3} = 60°$ $\quad$ $45° \times \frac{\pi \text{rad}}{180°} = \frac{45\pi \text{rad}}{180} = \frac{\pi}{4}\text{rad}$

Note: if the number does not have a degree symbol ° then assume its rad
1 revolution is 2π rad

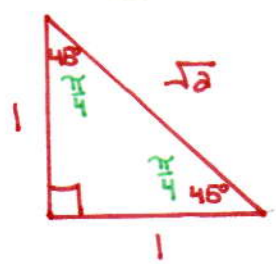
Reciprocal Trigonometric Ratios:

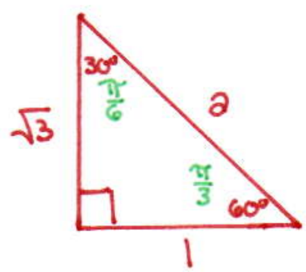
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

CAST Rule:

S Only sine is positive	A All are positive
T Only tangent is positive	C Only cosine is positive

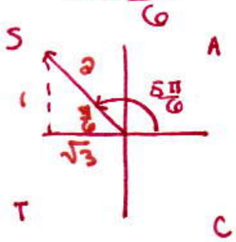
Special Triangles:





Using Special Triangles:

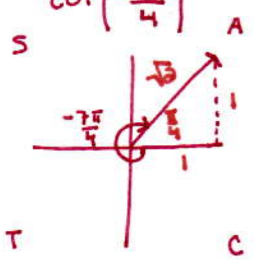
$\cos\frac{5\pi}{6}$



$R.A.A = \frac{\pi}{6}$

$\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

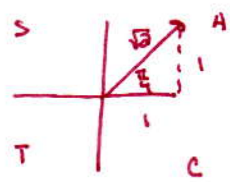
$\cot\left(-\frac{7\pi}{4}\right)$



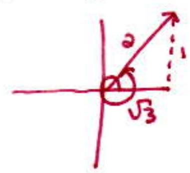
$R.A.A = \frac{\pi}{4}$

$\cot\left(\frac{\pi}{4}\right) = 1$

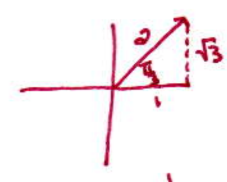
$\sin^2\frac{\pi}{4} + 2\sin\frac{13\pi}{6}\cot\frac{\pi}{3}$



$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$



$\sin\frac{13\pi}{6} = \frac{1}{2}$



$\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

$\sin^2\frac{\pi}{4} + 2\sin\frac{13\pi}{6}\cot\frac{\pi}{3}$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{1}{2} + \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{3}}{2\sqrt{3}} + \frac{2}{2\sqrt{3}}$

$= \frac{\sqrt{3}+2}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{3+2\sqrt{3}}{6}$

Properties of $y = \sin x$:

Domain: $x \in \mathbb{R}$ (Degrees + radians) max value: 1 Amplitude: 1

Range: $-1 \leq y \leq 1$ (Degrees + radians) min value: -1 Asymptotes: none

Period: $360°$ or 2π rad y-intercept: 0 x-intercepts: $180n, n \in \mathbb{Z}$ or $\pi n, n \in \mathbb{Z}$

Properties of $y = \cos x$:

Domain: $x \in \mathbb{R}$ (Degrees + radians) max value: 1 Amplitude: 1

Range: $-1 \leq y \leq 1$ (Degrees + radians) min value: -1 Asymptote: none

Period: $360°$ or 2π rad y-intercept: 1 x-intercepts: $90° + 180°n, n \in \mathbb{Z}$ or $\frac{\pi}{2} + \pi n, n \in \mathbb{Z}$

Properties of $y = \tan x$:

Domain: $x \in \mathbb{R} \mid x \neq 90 + 180n, n \in \mathbb{Z}$ (Degrees)
$x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$ (Radians)

Range: $y \in \mathbb{R}$ (Degrees + Radians)
Period: π

Absolute max or min value: none local max or min value: none Asymptotes: $90 + 180n$ $n \in \mathbb{Z}$
$\frac{\pi}{2} + \pi n$ $n \in \mathbb{Z}$

x-intercepts: $180n, n \in \mathbb{Z}$ or $\pi n, n \in \mathbb{Z}$ y-intercept: 0 (degrees + rad)

Transformations of Trigonometric Functions:

Applying transformations to sine, cos and tan graphs are the same like any other function. To find the period of these graphs an easily formula is $P = \frac{2\pi}{k}$. To find the k value is to rearrange the formula to $k = \frac{2\pi}{P}$. To find key points on these graphs you divide the period by 4.

Properties of $y = \csc x$:

Domain: $x \in \mathbb{R} \mid x \neq k\pi$ $k \in \mathbb{Z}$ Range: $y \in \mathbb{R} \mid y \leq -1$ and $y \geq 1$ Absolute max + min value: none

Period: 2π Local max + min value: -1 and 1 x-intercepts + y-intercepts: none

Asymptotes: $x = k\pi, k \in \mathbb{Z}$

Properties of $y = \sec x$:

Domain: $x \in \mathbb{R} \mid x \neq k\frac{\pi}{2}, k \in \mathbb{Z}$ Range: $y \in \mathbb{R} \mid y \leq -1$ and $y \geq 1$ Absolute max + min value: none

Period: 2π Local max + min value: -1 and 1 x-intercepts: none y-intercept: $y = 1$

Asymptotes: $x = k\frac{\pi}{2}, k \in \mathbb{Z}$

Properties of $y = \cot x$:

Domain: $x \in \mathbb{R} \mid x \neq k\pi$ $k \in \mathbb{Z}$ Range: $y \in \mathbb{R}$ Absolute max + min value: none

Local max + min value: none Period: π X-intercept: $k\frac{\pi}{2}, k \in \mathbb{Z}$ y-intercept: $y = 0$

Asymptote: $x = k\pi, k \in \mathbb{Z}$

Rates of Change of Trigonometric Functions:

The process of average rate of change and instantaneous rate of change is the same when concerning trigonometric functions.

Unit 7 - Trigonometric Identities and Equations

Equivalent Trigonometric Expressions:

$\sin(-x) = -\sin x \quad \cos(-x) = \cos x \quad \tan(-x) = -\tan x$

$\sin(\frac{\pi}{2} - x) = \cos x \quad \cos(\frac{\pi}{2} - x) = \sin x \quad \tan(\frac{\pi}{2} - x) = \cot x$

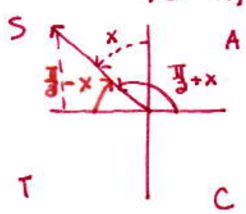
Note: the above expressions must be memorized

Steps to solving trigonometric functions:

1. Draw the angle
2. Find the related acute angle
3. Express the ratio given in terms of the related acute angle
4. Watch for sign change

Express as a trigonometric function of the acute angle x

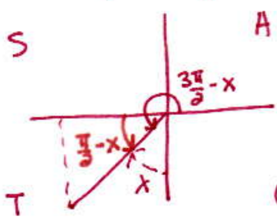
$\tan(\frac{\pi}{2} + x)$



R.A.A $= \frac{\pi}{2} - x$

$\tan(\frac{\pi}{2} - x) = -\cot x$

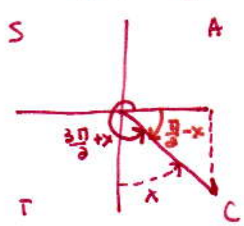
$\sin(\frac{3\pi}{2} - x)$



R.A.A $= \frac{\pi}{2} - x$

$\sin(\frac{\pi}{2} - x) = -\cos x$

$\cos(\frac{3\pi}{2} + x)$



R.A.A $= \frac{\pi}{2} - x$

$\cos(\frac{\pi}{2} - x) = -\sin x$

Addition and Subtraction Formulas:

Drive formulas for the sine and cosine of the sum and difference of two angles

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Replacing β with $-\beta$ we have:

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$

To develope formulas for $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$ we:

Let $\alpha = 90° - \alpha$ Let $\beta = -\beta$

$$\begin{aligned}\cos[(90°-\alpha)-\beta] &= \cos(90°-\alpha)\cos(-\beta) + \sin(90°-\alpha)\sin(-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha(-\sin\beta)\end{aligned}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos(-\beta) + \sin(-\beta)\cos\alpha\\ &= \sin\alpha\cos\beta - \sin\beta\cos\alpha\end{aligned}$$

$\sin(90°-x) = \cos x$
$\cos(90°-x) = \sin x$
$\tan(90°-x) = \cot x$

Finding a formula for $\tan(\alpha+\beta)$:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\end{aligned}$$

Divide the numerator and denominator by $\cos\alpha\cos\beta$

$$\begin{aligned}&= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \times \frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}\\ &= \frac{\frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha}\cancel{\cos\beta}}{\cancel{\cos\alpha}\cancel{\cos\beta}} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\end{aligned}$$

Let $\beta = -\beta$

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}\\ &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\end{aligned}$$

Note: all of these formulas are given to you on the trig tool box, you only need to know these proofs

Double Angle Formulas:

Using addition formulas of sine:

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Let $\beta = \alpha$

$\sin(2\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$= 2\sin\alpha\cos\alpha$

Using addition formulas of cosine:

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Let $\beta = \alpha$

$\cos(2\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$= \cos^2\alpha - \sin^2\alpha$

More double angle formulas of cosine:

$\cos(2\alpha) = \cos^2\alpha - (1-\cos^2\alpha)$

$= \cos^2\alpha - 1 + \cos^2\alpha$

$= 2\cos^2\alpha - 1$

$\cos(2\alpha) = (1-\sin^2\alpha) - \sin^2\alpha$

$= 1 - \sin^2\alpha - \sin^2\alpha$

$= 1 - 2\sin^2\alpha$

$\sin^2 x + \cos^2 x = 1$

$\sec^2 x = 1 + \tan^2 x$

$\csc^2 x = 1 + \cot^2 x$

Using addition formulas of tan:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Let $\beta = \alpha$

$$\tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Golden Guide To Proving Identities:

① If one side of the identity has only certain types of ratios make the other side only have those types of ratios (often sin and cos)

② If one side has only a single fraction term use a common denominator to make the other side have only one fraction term

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$ $\quad$ $\cos^2 x = 1 - \sin^2 x$

③ Make all the arguements x ($\sin x, \cos x, \tan x$ not $\sin(x+y), \cos(x+y), \tan(x+y)$)

④ FACTOR! i.e - $1 - \cos^2 x = (1+\cos x)(1+\cos x)$

$2\cos^2 x + \cos x = \cos x(2\cos x + 1)$

Example 1: Prove $\cos(x+y)\cos(x-y) = \cos^2 x + \cos^2 y - 1$

L.S	R.S
$\cos(x+y)\cos(x-y)$	$\cos^2 x + \cos^2 y - 1$

L.S

$\cos(x+y)\cos(x-y)$

$= (\cos x\cos y - \sin x\sin y)(\cos x\cos y + \sin x\sin y)$

$= \cos^2 x\cos^2 y + \sin x\cos x\sin y\cos y - \sin x\cos x\sin y\cos y - \sin^2 x\sin^2 y$

$= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y$

$= \cos^2 x\cos^2 y - (1-\cos^2 x)(1-\cos^2 y)$

$= \cos^2 x\cos^2 y - (1 - \cos^2 y - \cos^2 x + \cos^2 x\cos^2 y)$

$= -1 + \cos^2 y + \cos^2 x$

$= \cos^2 x + \cos^2 y - 1$

R.S

$\cos^2 x + \cos^2 y - 1$

LS = RS ∴ True

Example 2: Prove $\frac{\sin 2x}{1-\cos 2x} = 2\csc 2x - \tan x$

L.S

$\frac{\sin 2x}{1-\cos 2x}$

$= \frac{2\sin x\cos x}{1-(\cos^2 x - \sin^2 x)}$

$= \frac{2\sin x\cos x}{1-\cos^2 x + \sin^2 x}$

$= \frac{\cancel{2}\cancel{\sin x}\cos x}{\cancel{2}\sin^{\cancel{2}} x}$

$= \frac{\cos x}{\sin x}$

$= \frac{\cos x}{\sin x}$

R.S

$2\csc 2x - \tan x$

$= 2\frac{1}{\sin 2x} - \frac{\sin x}{\cos x}$

$= \cancel{2}\frac{1}{\cancel{2}\sin x\cos x} - \frac{\sin x}{\cos x}$

$= \frac{1}{\sin x\cos x} - \frac{\sin x}{\sin x\cos x}$

$= \frac{1-\sin^2 x}{\sin x\cos x}$

$= \frac{\cos^{\cancel{2}} x}{\sin x\cancel{\cos x}}$

$= \frac{\cos x}{\sin x}$

LS = RS ∴ True

Solving Linear Trigonometric Equations:

When solving linear trigonometric equations you are finding the x value involving sin, cos tan, sec, csc, cot. They have an equal sign no squared, cubed or square roots.

Note: You can solve these equation by drawing their graphs or using the special triangles

Steps to solving linear trig equations:

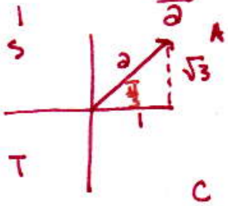
1. Find the related acute angle
2. Use the negative to determine which quadrants the terminal arm could lie in
3. Draw the terminal arm and the related acute angle (sketch it)
4. Using the domain state the values of x that fit that criteria

Solve the following equations for $0° \leq \theta \leq 360°$ (or $0 \leq \theta \leq 2\pi$)

$\sin\theta = \frac{\sqrt{3}}{2}$

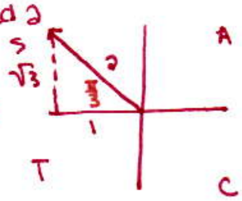
quad 1

R.A.A $= \frac{\pi}{3}$



quad 2

R.A.A $= \frac{\pi}{3}$



$\therefore\ x = \frac{\pi}{3}$

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\csc\theta = -1.7013$

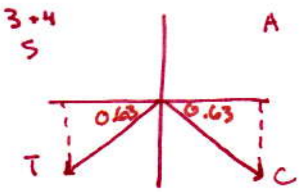
$\frac{1}{\sin\theta} = -1.7013$

$\sin\theta = \frac{1}{1.7013}$

R.A.A $= \sin^{-1}\left(\frac{1}{1.7013}\right)$

$= 0.63$

quad 3 + 4



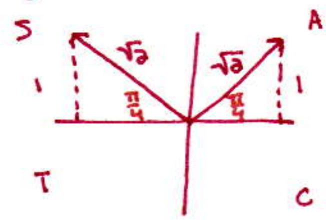
$\theta = \pi + 0.63°$ $\quad$ $\theta = 2\pi - 0.63°$

$= 3.77\text{rad}$ $\quad$ $= 5.65\text{rad}$

$\sqrt{2}\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{\sqrt{2}}$

quad 1 + 2 $\quad$ R.A.A $= \frac{\pi}{4}$



$\theta = \frac{\pi}{4}$ $\quad$ $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Trig Toolbox

Reciprocal Identities:

$$\csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \cot x = \frac{1}{\tan x}$$

Quotient Identities:

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1 \quad \sec^2 x = 1 + \tan^2 x \quad \csc^2 x = 1 + \cot^2 x$$

Addition + Subtraction Formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double Angle Formulas:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2\cos^2 x - 1$$

$$= 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Unit 8 - Logarithms

Logarithmic Function - where the input is the base and the output is the exponent

a log is an exponent, it is the inverse of an exponential

Equation of a Log:

$$\log_a x$$

The expression $\log_a x$ means the exponent that must be applied to base a to obtain the value of x. Also know as the logarthmic form.

Exponential Form:

$$x = a^y$$

Write the inverse of $f(x) = 4^x$ in logarthmic form

$f(x) = 4^x$

$y = 4^x$

for inverse:

$x = 4^y$ the exponent you put on 4 to get x is y

$y = \log_4 x$

$\therefore$ the inverse is $f^{-1}(x) = \log_4 x$

Write $y = \log_3 x$ in exponential form

$y = \log_3 x$

$3^y = x$ y is the exponent you applu to 3 to get x

$x = 3^y$

Graphing Logarithmic Functions:

Steps:

1. Change the function from logarithmic to exponential
2. Graph the exponential function
3. Applu all the transformations to the graph

Common Logarithms:

$$\log x = \log_{10} x$$

Important Properties of Logarithms:

$$\log_a 1 = 0 \qquad \log_a a^x = x \qquad a^{\log_a x} = x$$

Log Laws:

There are three laws; logarithm product law, logarithm quotient law, logarithm power of a power law.

$$\log_a mn = \log_a m + \log_a n \qquad \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n \qquad \log_a m^y = y\log_a m$$

Solving Exponential Equations

Solve $10(3)^x = 2430$

$$3^x = \frac{2430}{10}$$
$$3^x = 243$$
$$3^x = 3^5$$
$$\therefore x = 5$$

$$3^{x+2} - 3^x = 216$$
$$3^x(3^2 - 1) = 216$$
$$3^x(8) = 216$$
$$3^x = 27$$
$$3^x = 3^3$$
$$\therefore x = 3$$

$$3^x = 100$$
$$\log 3^x = \log 100$$
$$x\log 3 = \log 100$$
$$x = \frac{\log 100}{\log 3}$$
$$\therefore x = 4.192$$

$$2^{x+1} = 3^{x-1}$$
$$\log 2^{x+1} = \log 3^{x-1}$$
$$(x+1)\log 2 = (x-1)\log 3$$
$$x\log 2 + \log 2 = x\log 3 - \log 3$$
$$x\log 2 - x\log 3 = -\log 3 - \log 2$$
$$x(\log 2 - \log 3) = -\log 3 - \log 2$$
$$x = \frac{-\log 3 - \log 2}{\log 2 - \log 3}$$
$$= 4.419$$

Evaluating Logarithms With A Base Other Than 10:

$$\log_a x = \frac{\log x}{\log a}$$

Richter Scale:

You subtract the magnitudes then you put that number ten to that number

i.e - $7 - 5.5 = 1.5$ $10^{1.5} = 31.6$

Sound Intensity:

You subtract the decibles then divide that number by ten and finally set that number to the power of ten.

i.e - $125 - 53 = 72$ $72 \div 10 = 7.2$ $10^{7.2} = 15.8$ million

pH Scale:

$$pH = -\log[H^+]$$

Exponential Models:

Any situation that involves repeated multiplication can be modelled using:

$$y = c\,(a^x)$$

Final Initial Growth Factor

- Doubling : 2
- Half Life : $\frac{1}{2}$
- Growth by 12% : 1.12
- Decay by 25% : 0.75

Compound Intrest Formula:

concerns about money problems

$$A = P(1+i)^n$$

n ← number of compounds

Final amount, Prinicpal (Initial), 1 + interest rate per period

- quarterly - 4
- monthly - 12
- weekley - 52
- daily - 365

Example: Mr. Todd needs \$5000. He invests \$3000 at 18% compounded semi annually. How long will it take to become \$5000?

$A = P(1+i)^n$

$5000 = 3000(1 + \frac{0.18}{2})^n$

$\frac{5000}{3000} = (1 + 1.09)^n$

$\frac{5}{3} = 1.09^n$

$\log\frac{5}{3} = \log 1.09^n$

$\log\frac{5}{3} = n\log 1.09$

$\frac{\log\frac{5}{3}}{\log 1.09} = n$

$n = 5.93$ semi anually

$\frac{5.93}{2} = 2.96$ years

$\therefore$ 3 years