The Derivatives of Exponential Functions

We will begin our investigation of the derivatives of exponential functions by considering a very special exponential function:

$$f(x) = e^x$$

To find the derivative of this function, we will use our definition of the derivative.

Here we go!

$$f(x) = e^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x\left(e^h - 1\right)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

To find $\lim_{h \to 0} \frac{e^h - 1}{h}$, we can use tables.

h	$\frac{e^h - 1}{h}$
0.1	1.051709181
0.01	1.005016708
0.001	1.000500167
0.0001	1.000050002

h	$\frac{e^h - 1}{h}$
-0.1	0.951625819
-0.01	0.995016625
-0.001	0.999500166
-0.0001	0.999950001

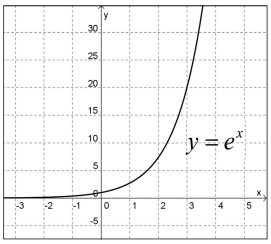
It appears that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$

So, we have

$$f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$f'(x) = e^x(1)$$

$$f'(x) = e^x$$



So, the derivative of e^x is e^x!

In other words, at any point on the graph of $y = e^x$, the tangent slope is the same as the *y*-value!

Question for Discussion

Are there any other functions that are their own derivative?

$$\frac{d}{dx} e^x = e^x$$

Some examples (complete on a separate page):

Differentiate each of the following functions (with respect to *x*).

a) $f(x) = e^{2x}$ b) $f(x) = e^{x^2}$ c) $f(x) = xe^x$

d) $f(x) = x^4 e^{-x}$ e) $f(x) = \frac{x^3}{e^x}$ f) $f(x) = \frac{e^{x^2}}{3x^3 + 4x}$

When it comes to taking the derivative of exponential functions, we have only considered one special exponential function…

$$f(x) = e^x$$

This function has the special property that it is its own derivative!

This property, however, is not true for exponential functions in general.

Now, let's consider the derivative of the general exponential function

$f(x) = a^x$, where $a > 0$ and $a \neq 1$

Derivation

Using our knowledge of the derivative of $y = e^x$ and the fact that $a^x = e^{x \ln a}$, determine the derivative of the general exponential function $f(x) = a^x$ (on a separate page).

In general,

If $f(x) = a^x$, where $a > 0$, then $f'(x) = a^x \ln a$.

Some examples…

1) Differentiate each of the following.

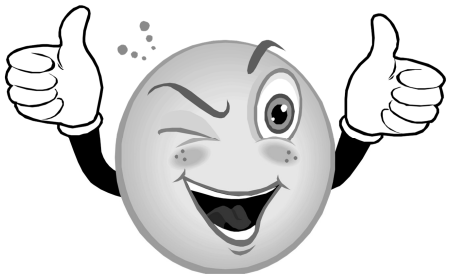
a) $f(x) = 5^x$

$f'(x) = 5^x \ln 5$

b) $f(x) = 5^{2x-3}$

$f'(x) = 5^{2x-3}(\ln 5)(2)$

$= 2\left(5^{2x-3}\right)\ln 5$

2) At what point on the graph of the function $y = 2^t - 3$ does the tangent line have a slope of 21? (*complete on a separate page*)

Don't forget, we may have to use the product rule and the quotient rule with questions involving the derivative of exponential functions!