PART A

1. What rules do you know for calculating derivatives? Give examples of each rule.

2. Determine $f'(x)$ for each of the following functions:

a. $f(x) = 4x - 7$

b. $f(x) = x^3 - x^2$

c. $f(x) = -x^2 + 5x + 8$

d. $f(x) = \sqrt[3]{x}$

e. $f(x) = \left(\frac{x}{2}\right)^4$

f. $f(x) = x^{-3}$

3. Differentiate each function. Use either Leibniz notation or prime notation, depending on which is appropriate.

a. $h(x) = (2x + 3)(x + 4)$

b. $f(x) = 2x^3 + 5x^2 - 4x - 3.75$

c. $s = t^2(t^2 - 2t)$

d. $y = \frac{1}{5}x^5 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 1$

e. $g(x) = 5(x^2)^4$

f. $s(t) = \frac{t^5 - 3t^2}{2t}, t > 0$

4. Apply the differentiation rules you learned in this section to find the derivatives of the following functions:

a. $y = 3x^{\frac{5}{3}}$

b. $y = 4x^{-\frac{1}{2}} - \frac{6}{x}$

c. $y = \frac{6}{x^3} + \frac{2}{x^2} - 3$

d. $y = 9x^{-2} + 3\sqrt{x}$

e. $y = \sqrt{x} + 6\sqrt{x^3} + \sqrt{2}$

f. $y = \frac{1 + \sqrt{x}}{x}$

PART B

5. Let s represent the position of a moving object at time t. Find the velocity $v = \frac{ds}{dt}$ at time t.

a. $s = -2t^2 + 7t$

b. $s = 18 + 5t - \frac{1}{3}t^3$

c. $s = (t - 3)^2$

6. Determine $f'(a)$ for the given function $f(x)$ at the given value of a.

a. $f(x) = x^3 - \sqrt{x}, a = 4$

b. $f(x) = 7 - 6\sqrt{x} + 5x^{\frac{2}{3}}, a = 64$

7. Determine the slope of the tangent to each of the curves at the given point.

a. $y = 3x^4, (1, 3)$

b. $y = \frac{1}{x^{-5}}, (-1, -1)$

c. $y = \frac{2}{x}, (-2, -1)$

d. $y = \sqrt{16x^3}, (4, 32)$

8. Determine the slope of the tangent to the graph of each function at the point with the given x-coordinate.

a. $y = 2x^3 + 3x, x = 1$

b. $y = 2\sqrt{x} + 5, x = 4$

c. $y = \frac{16}{x^2}, x = -2$

d. $y = x^{-3}(x^{-1} + 1), x = 1$

9. Write an equation of the tangent to each curve at the given point.

a. $y = 2x - \frac{1}{x}, P(0.5, -1)$

b. $y = \frac{3}{x^2} - \frac{4}{x^3}, P(-1, 7)$

c. $y = \sqrt{3x^3}, P(3, 9)$

d. $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right), P(1, 2)$

e. $y = (\sqrt{x} - 2)(3\sqrt{x} + 8), P(4, 0)$

f. $y = \frac{\sqrt{x} - 2}{\sqrt[3]{x}}, P(1, -1)$

10. What is a normal to the graph of a function? Determine the equation of the normal to the graph of the function in question 9, part b., at the given point.

11. Determine the values of x so that the tangent to the function $y = \frac{3}{\sqrt[3]{x}}$ is parallel to the line $x + 16y + 3 = 0$.

12. Do the functions $y = \frac{1}{x}$ and $y = x^3$ ever have the same slope? If so, where?

13. Tangents are drawn to the parabola $y = x^2$ at $(2, 4)$ and $\left(-\frac{1}{8}, \frac{1}{64}\right)$. Prove that these lines are perpendicular. Illustrate with a sketch.

14. Determine the point on the parabola $y = -x^2 + 3x + 4$ where the slope of the tangent is 5. Illustrate your answer with a sketch.

15. Determine the coordinates of the points on the graph of $y = x^3 + 2$ at which the slope of the tangent is 12.

16. Show that there are two tangents to the curve $y = \frac{1}{5}x^5 - 10x$ that have a slope of 6.

17. Determine the equations of the tangents to the curve $y = 2x^2 + 3$ that pass through the following points:

a. point $(2, 3)$

b. point $(2, -7)$

18. Determine the value of a, given that the line $ax - 4y + 21 = 0$ is tangent to the graph of $y = \frac{a}{x^2}$ at $x = -2$.

19. It can be shown that, from a height of h metres, a person can see a distance of d kilometres to the horizon, where $d = 3.53\sqrt{h}$.

a. When the elevator of the CN Tower passes the 200 m height, how far can the passengers in the elevator see across Lake Ontario?

b. Find the rate of change of this distance with respect to height when the height of the elevator is 200 m.

20. An object drops from a cliff that is 150 m high. The distance, d, in metres, that the object has dropped at t seconds in modelled by $d(t) = 4.9t^2$.

a. Find the average rate of change of distance with respect to time from 2 s to 5 s.

b. Find the instantaneous rate of change of distance with respect to time at 4 s.

c. Find the rate at which the object hits the ground to the nearest tenth.

21. A subway train travels from one station to the next in 2 min. Its distance, in kilometres, from the first station after t minutes is $s(t) = t^2 - \frac{1}{3}t^3$. At what times will the train have a velocity of 0.5 km/min?

22. While working on a high-rise building, a construction worker drops a bolt from 320 m above the ground. After t seconds, the bolt has fallen a distance of s metres, where $s(t) = 5t^2, 0 \leq t \leq 8$. The function that gives the height of the bolt above ground at time t is $R(t) = 320 - 5t^2$. Use this function to determine the velocity of the bolt at $t = 2$.

23. Tangents are drawn from the point $(0, 3)$ to the parabola $y = -3x^2$. Find the coordinates of the points at which these tangents touch the curve. Illustrate your answer with a sketch.

24. The tangent to the cubic function that is defined by $y = x^3 - 6x^2 + 8x$ at point $A(3, -3)$ intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.

25. a. Find the coordinates of the points, if any, where each function has a horizontal tangent line.

i. $f(x) = 2x - 5x^2$

ii. $f(x) = 4x^2 + 2x - 3$

iii. $f(x) = x^3 - 8x^2 + 5x + 3$

b. Suggest a graphical interpretation for each of these points.

PART C

26. Let $P(a, b)$ be a point on the curve $\sqrt{x} + \sqrt{y} = 1$. Show that the slope of the tangent at P is $-\sqrt{\frac{b}{a}}$.

27. For the power function $f(x) = x^n$, find the x-intercept of the tangent to its graph at point $(1, 1)$. What happens to the x-intercept as n increases without bound $(n \rightarrow +\infty)$? Explain the result geometrically.

28. For each function, sketch the graph of $y = f(x)$ and find an expression for $f'(x)$. Indicate any points at which $f'(x)$ does not exist.

a. $f(x) = \begin{cases} x^2, x < 3 \\ x + 6, x \geq 3 \end{cases}$ b. $f(x) = |3x^2 - 6|$ c. $f(x) = ||x| - 1|$

ANSWERS

1. Answers may vary. For example:
constant function rule: $\frac{d}{dx}(5) = 0$
power rule: $\frac{d}{dx}(x^3) = 3x^2$
constant multiple rule: $\frac{d}{dx}(4x^3) = 12x^2$
sum rule: $\frac{d}{dx}(x^2 + x) = 2x + 1$
difference rule: $\frac{d}{dx}(x^3 - x^2 + 3x) = 3x^2 - 2x + 3$

2. **a.** 4 **d.** $\frac{1}{3\sqrt[3]{x^2}}$
b. $3x^2 - 2x$ **e.** $\frac{x^3}{4}$
c. $-2x + 5$ **f.** $-3x^{-4}$

3. **a.** $4x + 11$ **d.** $x^4 + x^2 - x$
b. $6x^2 + 10x - 4$ **e.** $40x^7$
c. $4t^3 - 6t^2$ **f.** $2t^3 - \frac{3}{2}$

4. **a.** $5x^{\frac{2}{3}}$
b. $-2x^{-\frac{3}{2}} + 6x^{-2}$
c. $\frac{-18}{x^4} - \frac{4}{x^3}$
d. $-18x^{-3} + \frac{3}{2}x^{-\frac{1}{2}}$
e. $\frac{1}{2}(x^{-\frac{1}{2}}) + 9x^{\frac{1}{2}}$
f. $-x^{-2} - \frac{1}{2}x^{-\frac{3}{2}}$

5. **a.** $-4t + 7$ **b.** $5 - t^2$ **c.** $2t - 6$

6. **a.** 47.75 **b.** $\frac{11}{24}$

7. **a.** 12 **c.** $-\frac{1}{2}$
b. 5 **d.** 12

8. **a.** 9 **c.** 4
b. $\frac{1}{2}$ **d.** -7

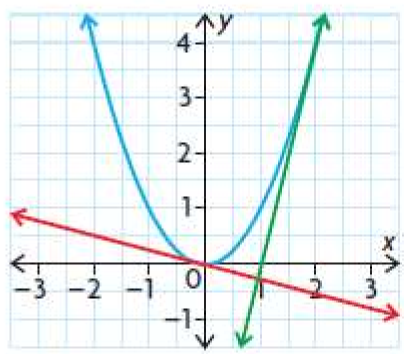
9. **a.** $6x - y - 4 = 0$
b. $18x - y + 25 = 0$
c. $9x - 2y - 9 = 0$
d. $x + y - 3 = 0$
e. $7x - 2y - 28 = 0$
f. $5x - 6y - 11 = 0$

10. A normal to the graph of a function at a point is a line that is perpendicular to the tangent at the given point;
$x + 18y - 125 = 0$

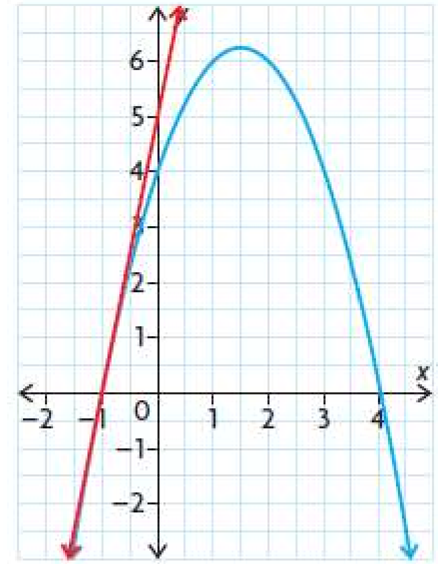
11. 8

12. no

13. $y = x^2, \frac{dy}{dx} = 2x$
The slope of the tangent at $A(2, 4)$ is 4 and at $B\left(-\frac{1}{8}, \frac{1}{64}\right)$ is $-\frac{1}{4}$.
Since the product of the slopes is -1, the tangents at $A(2, 4)$ and $B\left(-\frac{1}{8}, \frac{1}{64}\right)$ will be perpendicular.



14. $(-1, 0)$



15. $(2, 10)$ and $(-2, -6)$

16. $y = \frac{1}{5}x^5 - 10x$, slope is 6
$\frac{dy}{dx} = x^4 - 10 = 6$
$x^4 = 16$
$x^2 = 4$ or $x^2 = -4$
$x = \pm 2$ non-real
Tangents with slope 6 are at the points $\left(2, -\frac{68}{5}\right)$ and $\left(-2, \frac{68}{5}\right)$.

17. **a.** $y - 3 = 0$; $16x - y - 29 = 0$
b. $20x - y - 47 = 0$; $4x + y - 1 = 0$

18. 7

19. **a.** 49.9 km
b. 0.12 km/m

20. a. 34.3 m/s

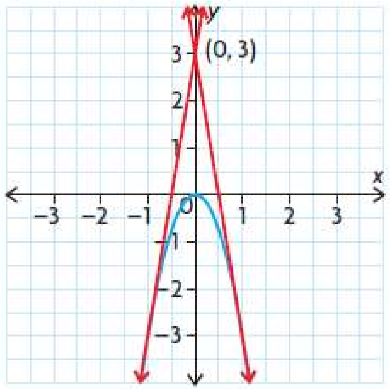
b. 39.2 m/s

c. 54.2 m/s

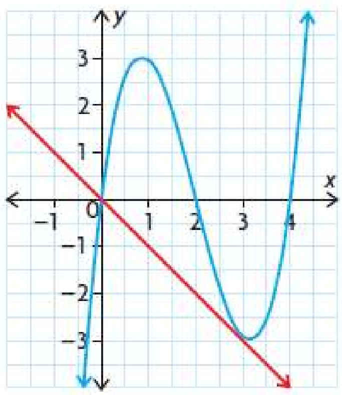
21. 0.29 min and 1.71 min

22. −20 m/s

23. $(1, -3)$ and $(-1, -3)$



24. $B(0, 0)$



25. a. i. $\left(\frac{1}{5}, \frac{1}{5}\right)$

ii. $\left(-\frac{1}{4}, -\frac{13}{4}\right)$

iii. $\left(\frac{1}{3}, \frac{103}{27}\right)$ and $(5, -47)$

b. At these points, the slopes of the tangents are zero, meaning that the rate of change of the value of the function with respect to the domain is zero. These points are also local maximum and minimum points.

26. $\sqrt{x} + \sqrt{y} = 1$

$P(a, b)$ is on the curve; therefore, $a \geq 0, b \geq 0$.

$$\sqrt{y} = 1 - \sqrt{x}$$
$$y = 1 - 2\sqrt{x} + x$$
$$\frac{dy}{dx} = -\frac{1}{2}\left(2x^{-\frac{1}{2}} + 1\right)$$

At $x = a$. Slope is

$$-\frac{1}{\sqrt{a}} + 1 = \frac{-1 + \sqrt{a}}{\sqrt{a}}.$$

But, $\sqrt{a} + \sqrt{b} = 1$

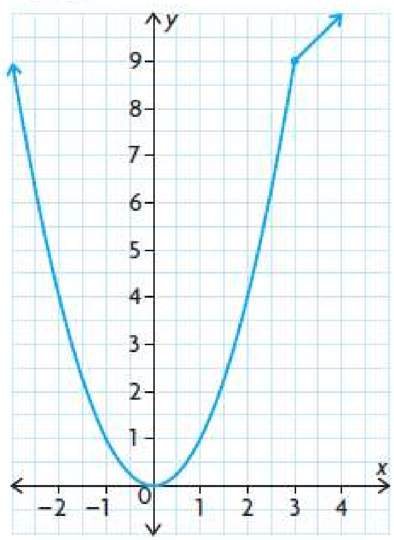
$$-\sqrt{b} = \sqrt{a} - 1$$

Therefore, slope is $-\frac{\sqrt{b}}{\sqrt{a}} = -\sqrt{\frac{b}{a}}$.

27. The x-intercept is $1 - \frac{1}{n}$, as $n \to \infty$, $\frac{1}{n} \to 0$, and the x-intercept approaches 1. As $n \to \infty$, the slope of the tangent at (1, 1) increase without bound, and the tangent approaches a vertical line having equation $x - 1 = 0$.

28. a. $f'(x) = \begin{cases} 2x, \text{ if } x < 3 \\ 1, \text{ if } x \geq 3 \end{cases}$

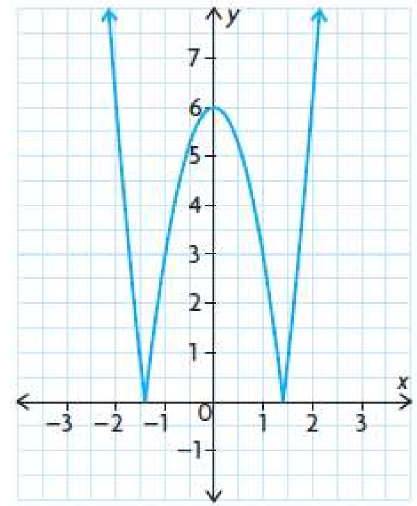
$f'(3)$ does not exist.



b.

$$f'(x) = \begin{cases} 6x, \text{ if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \\ -6x, \text{ if } -\sqrt{2} \leq x \leq \sqrt{2} \end{cases}$$

$f'(\sqrt{2})$ and $f'(-\sqrt{2})$ do not exist.



c. $f'(x) = \begin{cases} 1, \text{ if } x > 1 \\ -1, \text{ if } 0 < x < 1 \\ 1, \text{ if } -1 < x < 0 \\ -1, \text{ if } x < -1 \end{cases}$

$f'(0)$, $f'(-1)$, and $f'(1)$ do not exist.