

SOLVING QUADRATIC TRIGONOMETRIC EQUATIONS

In the previous lesson, the trigonometric equations that we solved were all *linear*. In this lesson, however, we will consider some nonlinear trigonometric equations, giving special attention to *quadratic trigonometric equations*.

When learning to solve trigonometric equations, it is often helpful to consider analogous equations that do not involve trigonometric expressions. Check out the following examples:

Trigonometric Equation	Similar to...
1) $\sin^2 x = \frac{1}{2}$	$x^2 = \frac{1}{2}$
2) $\tan^2 x - \tan x = 0$	$x^2 - x = 0$
3) $\cos^2 x + 3 \cos x + 2 = 0$	$x^2 + 3x + 2 = 0$
4) $3 \sin^2 x - \sin x - 2 = 0$	$3x^2 - x - 2 = 0$
5) $\sin x + 2 \sin x \cos x = 0$	$x + 2xy = 0$
6) $\cot x \cos x = \cot x$	$xy = x$
7) $\cos x \sin x = 0$	$xy = 0$
8) $\tan \theta \cos^2 \theta = \tan \theta$	$xy^2 = x$

Solve each of the above trigonometric equations on a separate page. Use degree measure with $0^\circ \leq x \leq 360^\circ$ for the first four equations and use radian measure with $0 \leq x \leq 2\pi$ for the final four equations. It may be helpful to use the non-trigonometric equations as a guide!

Sometimes it is necessary to use a trigonometric identity to help solve an equation...

Example

Solve the following equations (using degree measure) for $0^\circ \leq x \leq 360^\circ$.

a) $-10 \cos^2 x - 3 \sin x + 9 = 0$

b) $\sec^2 x + 5 \tan x = -2$

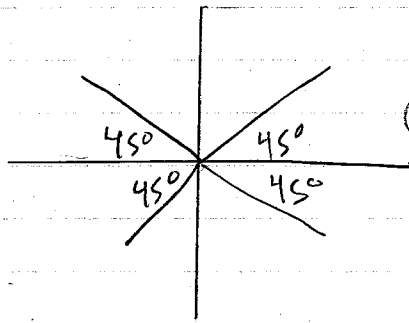
$0 \leq x \leq 2\pi$

$$1) \sin^2 x = \frac{1}{2}$$

$$\sin x = \pm \frac{1}{\sqrt{2}}$$

$$\text{R.A.A} = 45^\circ$$

Quadrant: I, II, III, IV



$$\therefore x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

$$2) \tan^2 x - \tan x = 0$$

$$\tan x (\tan x - 1) = 0$$

$$\tan x = 0$$

$$x = 0^\circ, 180^\circ, 360^\circ$$

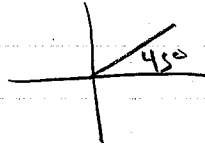
(from graph of $y = \tan x$)

$$\tan x - 1 = 0$$

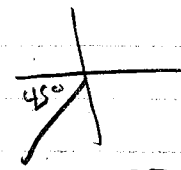
$$\tan x = 1$$

$$\text{R.A.A} = 45^\circ$$

Quadrants I + III



$$x = 45^\circ$$



$$x = 225^\circ$$

$$\therefore x = 0^\circ, 45^\circ, 180^\circ, 225^\circ, 360^\circ$$

$$3) \quad \cos^2 x + 3\cos x + 2 = 0$$

$$(\cos x + 2)(\cos x + 1) = 0$$

$$\cos x + 2 = 0$$

$$\cos x = -2$$

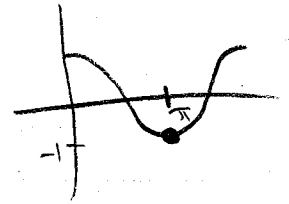
No solution
($\cos x$ never below -1)

$$\cos x + 1 = 0$$

$$\cos x = -1$$

$$x = \pi$$

(from graph) →



$$\therefore x = \pi$$

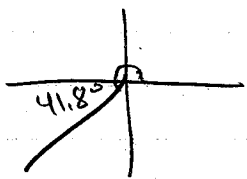
$$4) \quad 3\sin^2 x - \sin x - 2 = 0$$

$$(3\sin x + 2)(\sin x - 1) = 0$$

$$3\sin x + 2 = 0$$

$$\sin x = -\frac{2}{3}$$

R.A.A. = 41.8°
Quadrant III or IV

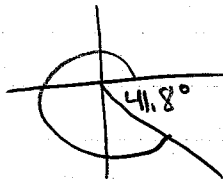


$$x = 221.8^\circ$$

$$\sin x - 1 = 0$$

$$\sin x = 1$$

$$x = 90^\circ \quad (\text{from sine graph})$$



$$x = 318.2^\circ$$

$$\therefore x = 90^\circ, 221.8^\circ, 318.2^\circ$$

$$5) \sin x + 2\sin x \cos x = 0$$

$$\sin x (1 + 2\cos x) = 0$$

$$\sin x = 0$$

$$x = 0, \pi, 2\pi$$

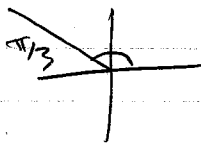
(from sine graph)

$$1 + 2\cos x = 0$$

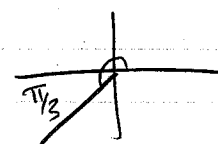
$$\cos x = -\frac{1}{2}$$

$$\text{R.A.A.} = \frac{\pi}{3}$$

Quadrant II or III



$$x = \frac{2\pi}{3}$$



$$x = \frac{4\pi}{3}$$

$$\therefore x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$$

$$6) \cot x \cos x = \cot x$$

$$\cot x \cos x - \cot x = 0$$

$$\cot x (\cos x - 1) = 0$$

$$\cot x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

(from $y = \cot x$ graph)

$$\cos x - 1 = 0$$

$$\cos x = 0$$

$$x = 0, 2\pi$$

(from $y = \cos x$ graph)

$$\therefore x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$$

$$7) \cos x \sin x = 0$$

$$\cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

(from $y = \cos x$ graph)

$$\sin x = 0$$

$$x = 0, \pi, 2\pi$$

(from $y = \sin x$ graph)

$$\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

$$8) \tan \theta \cos^2 \theta = \tan \theta$$
$$\tan \theta \cos^2 \theta - \tan \theta = 0$$
$$\tan \theta (\cos^2 \theta - 1) = 0$$

$$\tan \theta = 0$$

$$\theta = 0, \pi$$

(from \tan graph)

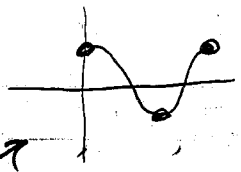
$$\cos^2 \theta - 1 = 0$$

$$\cos^2 \theta = 1$$

$$\cos \theta = \pm 1$$

$$\theta = 0, \pi, 2\pi$$

(from graph)



$$\therefore \theta = 0, \pi, 2\pi$$

$$a) \quad -10\cos^2 x - 3\sin x + 9 = 0$$

$$-10(1 - \sin^2 x) - 3\sin x + 9 = 0$$

$$-10 + 10\sin^2 x - 3\sin x + 9 = 0$$

$$10\sin^2 x - 3\sin x - 1 = 0$$

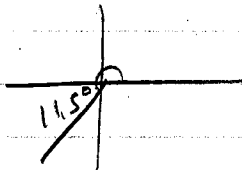
$$(5\sin x + 1)(2\sin x - 1) = 0$$

$$5\sin x + 1 = 0$$

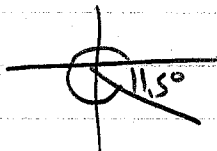
$$\sin x = -\frac{1}{5}$$

$$\text{R.A.A.} = 11.5^\circ$$

Quadrants III + IV



$$x = 191.5^\circ$$



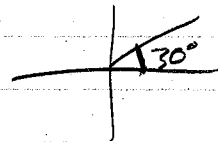
$$x = 348.5^\circ$$

$$2\sin x - 1 = 0$$

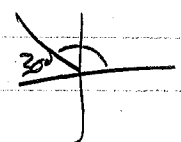
$$\sin x = \frac{1}{2}$$

$$\text{R.A.A.} = 30^\circ$$

Quadrants I + II



$$x = 30^\circ$$



$$x = 150^\circ$$

$$\therefore x = 30^\circ, 150^\circ, 191.5^\circ, 348.5^\circ$$

b)

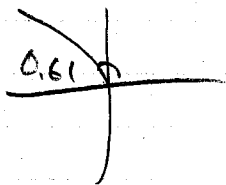
$$\begin{aligned} \sec^2 x + 5 \tan x &= -2 \\ (1 + \tan^2 x) + 5 \tan x &= -2 \\ \tan^2 x + 5 \tan x + 3 &= 0 \end{aligned}$$

$$\tan x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$

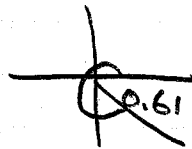
$$\tan x = -0.70 \quad \text{or}$$

$$\text{R.A.} = 0.61$$

Quadrant II + IV



$$\begin{aligned} x &= \pi - 0.61 \\ &= 2.53 \end{aligned}$$

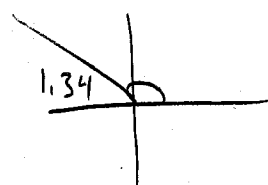


$$\begin{aligned} x &= 2\pi - 0.61 \\ &= 5.67 \end{aligned}$$

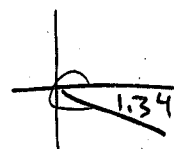
$$\tan x = -4.3$$

$$\text{R.A.} = 1.34$$

Quadrants II + IV



$$\begin{aligned} x &= \pi - 1.34 \\ &= 1.80 \end{aligned}$$



$$\begin{aligned} x &= 2\pi - 1.34 \\ &= 4.94 \end{aligned}$$

$$\therefore x = 1.80, 2.53, 4.94, 5.67$$