Creating Logarithmic Models

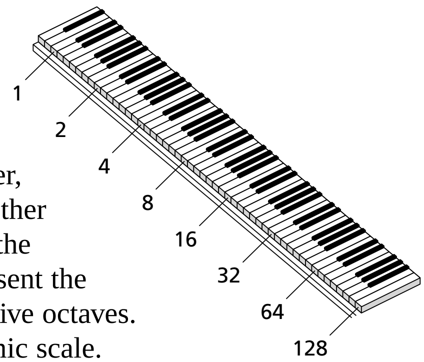
SETTING THE STAGE

Many quantities vary so greatly that the largest values are many millions of times greater than the smallest. The energy of earthquakes, the light received from stars, and the hydrogen ion concentration in acids and alkalis can be expressed as very large or very small values.

A very large scale would be needed to measure these quantities. So a scale based on exponents is used. Since exponents are logarithms, such scales are called **logarithmic scales**.

You are already familiar with linear scales. For instance, you have used a ruler, which is a linear scale, to measure distance. The intervals on a ruler are equally spaced, or uniform. The second interval is the same as the tenth interval.

In contrast, the intervals on a logarithmic scale are not equally spaced. A simple example of a logarithmic scale is a piano keyboard. The space on the keyboard between octaves is constant, because each octave has the same number of keys. However, the *frequency* of a note one octave above another is twice the original frequency. As shown in the diagram, the numbers 1, 2, 4, 8, 16, … represent the relative frequencies of the C notes in successive octaves. The sequence 1, 2, 4, 8, … forms a logarithmic scale. The logarithms to base 2 that correspond to the terms in this sequence are 0, 1, 2, 3, 4, … . The numbers in this second sequence are evenly spaced.



Music provides several examples of logarithmic scales. Have you wondered why the spacing between guitar frets is not even? Or why the scale on the tuner of an analog radio is not evenly spaced? These are examples of logarithmic scales.

What other scales are logarithmic? Do all logarithmic scales work in the same way?

In this section, you will investigate some common logarithmic scales.

EXAMINING THE CONCEPT

Working with Earthquakes and the Richter Scale

The amount of energy released by an earthquake can vary enormously. Some minor earthquakes release so little energy that people cannot feel them, while others release the same amount of energy as a billion tonnes of TNT. This range is so great that it is difficult to compare earthquakes on the basis of actual energy released. In 1935, Charles Richter solved this problem by developing the **Richter magnitude scale**. This open-ended scale uses logarithms to compare magnitude.

For example,

True Intensity	Richter Scale Magnitude
10^1	$\log_{10} 10^1 = 1$
10^4	$\log_{10} 10^4 = 4$
$10^{5.8}$	$\log_{10} 10^{5.8} = 5.8$

Suppose the difference between the magnitude of two earthquakes on the Richter scale is 2. Then the stronger earthquake is actually 10^2 or 100 times more intense. The table below compares the Richter magnitude for various examples and earthquakes.

Richter Magnitude	Example, or Epicentre of Earthquake
1.0	Large blast at a construction site
1.5	
2.0	Large quarry or mine blast
2.5	
3.0	
3.5	Less than 3.5: Earthquake generally not felt, but recorded
4.0	
4.5	Average tornado (total energy)
5.0	Quake often felt, but rarely causes damage
5.5	Little Skull Mountain, Nevada, 1992
6.0	Double Spring Flat, Nevada, 1994
6.5	Northridge, California, 1994
7.0	Hyogo-Ken Nanbu, Japan, 1995; Largest thermonuclear weapon
7.5	Landers, California, 1992
8.0	San Francisco, California, 1906
8.5	Anchorage, Alaska, 1964
9.0	Chilean earthquake, 1960
10.0	San-Andreas-type fault circling Earth
12.0	

Example 1 **Using the Richter Scale**

(a) How many more times intense, to the nearest whole number, was the 1995 earthquake in Hyogo-Ken Nanbu, Japan, that measured 7.0, than the 1992 earthquake in Little Skull Mountain, Nevada, that measured 5.5?

(b) How much more intense is an earthquake that measures 8 on the Richter scale than the average tornado?

Solution

(a) The Japanese earthquake measured 7 on the Richter scale. The Little Skull Mountain earthquake measured 5.5.

Find the difference between the magnitudes. $7 - 5.5 = 1.5$

Therefore, the difference in intensity is $10^{1.5}$.

$$10^{1.5} \doteq 31.6$$

Therefore, the 1995 earthquake in Japan was about 32 times more intense than the 1992 earthquake in Nevada.

(b) According to the table, the average tornado measures 4.5 on the Richter scale. To find the relative intensity of an earthquake measuring 8 on the Richter scale, find the difference. $8 - 4.5 = 3.5$

Therefore, the difference in intensity is $10^{3.5}$.

$$10^{3.5} \doteq 3162$$

Therefore, an earthquake measuring 8 on the Richter scale is 3162 times more intense than the average tornado.

EXAMINING THE CONCEPT

Measuring Sound Intensity

The loudness, L, of a sound is related to the sound's intensity, I, by $L = 10 \log \left(\frac{I}{I_0}\right)$, where L is measured in decibels (dB), I is measured in watts per square metre, and I_0 is the intensity of a barely audible sound. A barely audible sound has an intensity of $I_0 = 10^{-12}$ W/m^2. As a result, the loudness of a barely audible sound is calculated as follows:

$$\begin{aligned} L &= 10 \log \left(\frac{10^{-12}}{10^{-12}}\right) \\ L &= 10 \log 1 \\ L &= 10(0) \\ L &= 0 \text{ dB} \end{aligned}$$

The following table compares the actual loudness of some common sounds with their intensity measured in W/m^2.

Sound	Intensity (W/m^2)	Loudness (dB)	Ratio of Intensity to the Intensity at 0 dB $\left(\frac{I}{I_0}\right)$
threshold of sound	10^{-12}	0	1
rustling leaves	10^{-11}	10	10
whisper	10^{-10}	20	100
	10^{-9}	30	1 000
	10^{-8}	40	10 000
conversation	10^{-7}	50	100 000
light traffic	10^{-6}	60	1 000 000
heavy traffic	10^{-4}	80	100 000 000
boiler factory	10^{-2}	100	10 000 000 000
aircraft engine; loud music near amplifiers	10^{0}	120	100 000 000 000
painful to human ear	10^{1}	130+	1 000 000 000 000+

Notice that an increase from 10 dB to 20 dB is actually a tenfold increase in loudness. Loudness and intensity are often used interchangeably.

Example 2 Using the Sound Intensity Level Scale

How many times louder is a music concert of 125 dB than a conversation of 53 dB?

Noise pollution is a major threat in our industrialized world. There are very few pain receptors in our ears to warn us that gradual hearing loss may be occurring due to loud noise. Sounds over 90 dB, or even lower levels for prolonged, repeated periods of exposure, can cause permanent damage to the ear. You may have heard a "ringing" in your ears after a loud rock concert, which could signal permanent or temporary hearing loss.

Solution

The difference in the loudness of these two sounds is 72 dB. Substitute this value into the loudness formula.

$$L = 10 \log\left(\frac{I}{I_0}\right)$$

$$72 = 10 \log\left(\frac{I}{I_0}\right)$$

$$7.2 = \log\left(\frac{I}{I_0}\right)$$

$$10^{7.2} = \frac{I}{I_0}$$

The music concert is $10^{7.2}$ or about 15.8 million times as loud as the conversation.

EXAMINING THE CONCEPT

Other Logarithmic Scales

Psychologists use logarithmic scales to measure sense perceptions other than just sound. In chemistry, the pH — the measure of acidity or alkalinity of a substance — is based on a logarithmic scale.

The measure of the acidity of a solution is the number of hydrogen ions, H^+, in a specific volume of the solution. The number of hydrogen ions is measured in moles, and 1 mole $\doteq 6 \times 10^{23}$ particles. The hydrogen ion concentration, measured in moles of H^+ per litre, in most solutions is rather small. For example, the concentration of acid in the stomach is about 0.005 moles of H^+ per litre. Rather than deal with these small numbers, chemists use the **pH scale**, which ranges from 0 to 14.

$$\text{pH} = -\log(\text{concentration of H}^+)$$

A pH of 7 is neutral. The pH of an acidic solution ranges from 0 to less than 7. The pH of a basic solution ranges from more than 7 to 14.

Example 3 Using the pH Scale

Calculate the pH of each solution, to two decimal places, if necessary.

(a) concentration of $H^+ = 0.0001$

(b) concentration of $H^+ = 0.0022$

Solution

(a) $\text{pH} = -\log(\text{concentration of H}^+)$
$\text{pH} = -\log(0.0001)$
$= 4$

(b) $\text{pH} = -\log(\text{concentration of H}^+)$
$\text{pH} = -\log(0.0022)$
$\doteq 2.66$

CHECK, CONSOLIDATE, COMMUNICATE

1. Give examples of sequences of numbers that could be used for linear and logarithmic scales.
2. Why are logarithmic scales necessary?

KEY IDEAS

- The intervals on a linear scale are uniform, for example, 2, 4, 6, … . Each successive interval on a logarithmic scale is multiplied by the same factor. For example, in the sequence 2.5, 7.5, 22.5, … each term is multiplied by 3.
- Logarithmic scales are used to measure quantities that vary greatly. A logarithmic scale is formed from the exponents of the powers that represent measured quantities.