42

NAME: SOLUTIONS

KNOW /12	APP /12	INQ /12	COMM /6

MCV4UP - UNIT 2 – LIMITS AND CONTINUITY

TEST

GIVE ALL ANSWERS IN EXACT FORM, UNLESS STATED OTHERWISE.

PART A – This section is to be completed without the use of a calculator. Upon completing this section, hand it in to receive the remainder of the test.

1) Evaluate each of the following limits. All answers should be exact. (*K – 2,2,2,2,2,2 marks*)

a) $\lim\limits_{x\to 7}\dfrac{2x^2-11x-21}{x^2-3x-28}$

$= \lim\limits_{x\to 7}\dfrac{(2x+3)\cancel{(x-7)}}{(x+4)\cancel{(x-7)}}$

$= \dfrac{2(7)+3}{7+4}$

$= \dfrac{17}{11}$

b) $\lim\limits_{x\to -2}\dfrac{4-\sqrt{14-x}}{5x+10}\cdot\dfrac{(4+\sqrt{14-x})}{(4+\sqrt{14-x})}$

$= \lim\limits_{x\to -2}\dfrac{16-(14-x)}{5(x+2)(4+\sqrt{14-x})}$

$= \lim\limits_{x\to -2}\dfrac{\cancel{2+x}}{5\cancel{(x+2)}(4+\sqrt{14-x})}$

$= \dfrac{1}{5(4+\sqrt{14-(-2)})} = \dfrac{1}{40}$

c) $\lim\limits_{x\to 2}\begin{cases}3x^2+5, & x<2\\ x^3+5x-4, & x\ge 2\end{cases}\Bigg\} f(x)$

$\lim\limits_{x\to 2^-} f(x) = 17$

$\lim\limits_{x\to 2^+} f(x) = 14$

$\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$

$\therefore \lim\limits_{x\to 2} f(x)$ does not exist

d) $\lim\limits_{x\to -3}\dfrac{x^3+3x^2}{|x+3|}$

$= \lim\limits_{x\to -3}\dfrac{x^2(x+3)}{|x+3|}$

$= \lim\limits_{x\to -3}\begin{cases}-x^2, & x<-3\\ x^2, & x>-3\end{cases}\Bigg\} f(x)$

$\lim\limits_{x\to -3^-} f(x) = -9$, $\lim\limits_{x\to 3^+} f(x) = 9$

$\therefore \lim\limits_{x\to -3^-} f(x) \neq \lim\limits_{x\to -3^+} f(x)$

$\therefore \lim\limits_{x\to 3} f(x)$ does not exist

e) $\lim\limits_{x\to 4}\dfrac{\sqrt{x^2+7}+9}{3x-4x\sqrt{2^x}}$

$= \dfrac{\sqrt{4^2+7}+9}{3(4)-4(4)\sqrt{2^4}}$

$= \dfrac{\sqrt{23}+9}{-52}$

$= -\dfrac{\sqrt{23}+9}{52}$

f) $\lim\limits_{x\to 7}\dfrac{\sqrt[3]{x+57}-4}{x-7}$

Let $u = \sqrt[3]{x+57}$

$u^3-57 = x$

$= \lim\limits_{u\to 4}\dfrac{u-4}{u^3-57-7}$

$= \lim\limits_{u\to 4}\dfrac{u-4}{u^3-64}$

$= \lim\limits_{u\to 4}\dfrac{\cancel{u-4}}{\cancel{(u-4)}(u^2+4u+16)}$

$= \dfrac{1}{4^2+4(4)+16}$

$= \dfrac{1}{48}$

PART B – This section may be completed with the use of a calculator.

NAME: SOLUTIONS

2) Use the difference quotient method to determine the **equation** of the tangent to the graph of $f(x) = x^3 - 5x^2$ where $x = -1$. *(A – 4 marks)*

$f(-1) = -6$

$$m_{tan} = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{(-1+h)^3 - 5(-1+h)^2 - (-6)}{h}$$

$$= \lim_{h \to 0} \frac{h^3 - 3h^2 + 3h \cancel{-1} \cancel{-5} + 10h - 5h^2 \cancel{+6}}{h}$$

$$= \lim_{h \to 0} \frac{h^3 - 8h^2 + 13h}{h}$$

$$= \lim_{h \to 0} (h^2 - 8h + 13)$$

$$= 13$$

$y = 13x + b$

$-6 = 13(-1) + b$

$-6 = -13 + b$

$b = 7$

$\therefore y = 13x + 7$

3) A baseball thrown into the air has height given by the equation $h(t) = -5t^2 + 65t + 1$, where h is measured in metres and t is measured in seconds. Find the instantaneous velocity at $t = 6$ seconds. *(A – 4 marks)*

$h(6) =$

$$\text{IROC} = \lim_{k \to 0} \frac{h(6+k) - h(6)}{k}$$

$$= \lim_{k \to 0} \frac{-5(6+k)^2 + 65(6+k) + 1 - 211}{k}$$

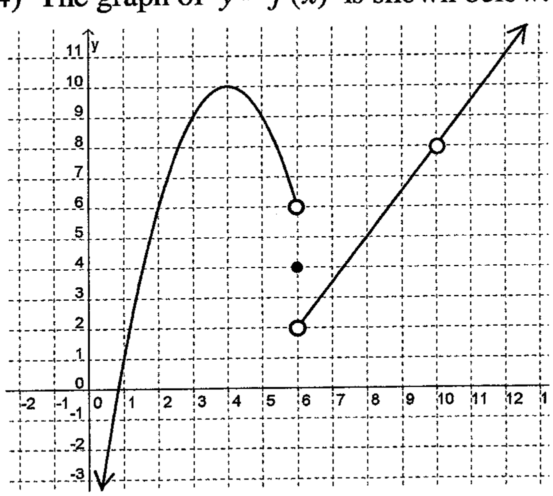
$$= \lim_{k \to 0} \frac{\cancel{-180} - 60k - 5k^2 \cancel{+390} + 65k \cancel{-210}}{k}$$

$$= \lim_{k \to 0} \frac{-5k^2 + 5k}{k}$$

$$= \lim_{k \to 0} (-5k + 5)$$

$= 5$ m/s.

4) The graph of $y = f(x)$ is shown below. Determine each of the following limits. *(A – 4 marks)*



a) $\lim_{x \to 6^+} f(x) =$ 2

b) $\lim_{x \to 4} f(x) =$ 10

c) $\lim_{x \to 10} f(x) =$ 8

d) $\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} =$ 0

5) Evaluate the following limits. (*I – 2 marks*)

a) $\lim_{x\to\infty}\dfrac{8x^2+7x-2}{2x^2+6x-4} = \underline{\ 4\ }$

b) $\lim_{x\to-\infty}\dfrac{5\sin x-4}{|x|} = \underline{\ 0\ }$

6) Evaluate $\lim_{x\to 0}\dfrac{x\csc(3x)+1}{x\csc(3x)}$. (*I – 2 marks*)

$$= \lim_{x\to 0}\frac{x\csc 3x}{x\csc 3x} + \lim_{x\to 0}\frac{1}{x\csc 3x}$$

$$= \lim_{x\to 0} 1 + \lim_{x\to 0}\frac{\sin 3x}{x}$$

$$= 1 + \lim_{x\to 0}\frac{3\sin 3x}{3x}$$

$$= 1 + 3\lim_{x\to 0}\frac{\sin 3x}{3x}$$

$$= 1 + 3(1)$$

$$= 4$$

7) Use the formal definition of a limit to prove that $\lim_{x\to 5}(\underbrace{2x-9}_{f(x)}) = 1$. (*I – 5 marks*)

We need to find $\delta > 0$ such that if $0 < |x-5| < \delta$, then $|f(x)-1| < \varepsilon$ for all $\varepsilon > 0$.

Now, $|f(x)-1| < \varepsilon$

$|2x-9-1| < \varepsilon$

$|2x-10| < \varepsilon$

$|2||x-5| < \varepsilon$

$2|x-5| < \varepsilon$

$|x-5| < \frac{\varepsilon}{2}$

∴ we can choose any δ in the interval $(0, \frac{\varepsilon}{2}]$

8) Determine the **coordinates** of all the points on the graph of $f(x) = x^2 + 5x - 7$ where the tangent is parallel to the line $y = 13x + 3$. (*I – 3 marks*)

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = 13$$

$$\lim_{h\to 0}\frac{(a+h)^2+5(a+h)-7-(a^2+5a-7)}{h} = 13$$

$$\lim_{h\to 0}\frac{\cancel{a^2}+2ah+h^2+\cancel{5a}+5h\cancel{-7}\cancel{-a^2}\cancel{-5a}\cancel{+7}}{h} = 13$$

$$\lim_{h\to 0}\frac{2ah+h^2+5h}{h} = 13$$

$$\lim_{h\to 0}(2a+h+5) = 13$$

$2a+5 = 13$

$2a = 8$

$a = 4$

$f(4) = 29$

∴ point is $(4, 29)$

9) Use the Sandwich Theorem to evaluate **ONE** of the following limits. (*C – 2 marks*)

a) $\lim\limits_{x\to-\infty} \dfrac{4e^x}{x^2+7}$

b) $\lim\limits_{x\to 0} x^4 \cos\dfrac{2}{x}$

For $x<0$,

$0<e^x<1$

$0<4e^x<4$

$0<\dfrac{4e^x}{x^2+7}<\dfrac{4}{x^2+7}$

$\lim\limits_{x\to-\infty} 0 = 0$ and $\lim\limits_{x\to-\infty} \dfrac{4}{x^2+7} = 0$

$\therefore \lim\limits_{x\to-\infty} \dfrac{4e^x}{x^2+7} = 0$

$\left|x^4\cos\frac{2}{x}\right| = |x^4|\left|\cos\frac{2}{x}\right|$

$= x^4\left|\cos\frac{2}{x}\right|$

$\leq x^4(1)$

$= x^4$

$\therefore -x^4 \leq x^4\cos\frac{2}{x} \leq x^4$

$\lim\limits_{x\to0}(-x^4)=0$ and $\lim\limits_{x\to0} x^4 = 0$

$\therefore \lim\limits_{x\to0} x^4\cos\frac{2}{x} = 0$

10) Ronald was working on a problem in which he was required to find an instantaneous rate of change for a function $f(x)$. The following equation appeared in his solution:

$$\text{Instantaneous rate of change} = \lim_{h\to0}\frac{3(5+h)^2-75}{h}$$

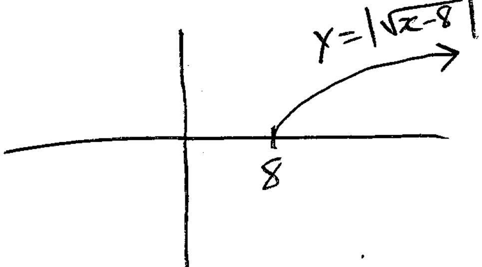
State two possible equations for Ronald's original function, $f(x)$. For each equation, state the point at which he would be finding the instantaneous rate of change. (*C – 2 marks*)

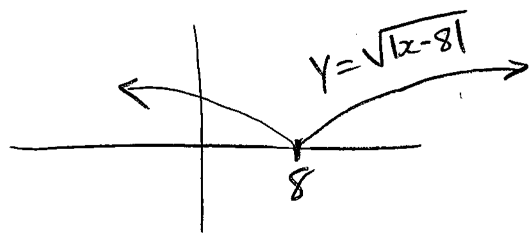
Equation: $f(x) = 3x^2$ Point: $(5, 75)$

Equation: $f(x) = 3x^2 - 10$ Point: $(5, 65)$

$\dfrac{3(5+h)^2-75}{h} = \dfrac{3(5+h)^2-10-65}{h}$

11) Paige and Turner were discussing limits. Turner claimed that $\lim\limits_{x\to8}\sqrt{|x-8|}$ is the same as $\lim\limits_{x\to8}\left|\sqrt{x-8}\right|$. Is Turner's claim correct? Explain. (*C – 2 marks*)





Turner's claim is incorrect. As shown above $\lim\limits_{x\to8}\left|\sqrt{x-8}\right|$ does not exist since the left-sided limit does not exist, whereas $\lim\limits_{x\to8}\sqrt{|x-8|} = 0$.